Question # 1 Show that

(i) $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ (ii) $s = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

Solution

(i) $\text{R.H.S} = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$= 4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}$

$= 4R\sqrt{\frac{(s-b)(s-c)(s-a)(s-c)(s-a)(s-b)}{(bc)(ac)(ab)}}$

$= 4R\sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$

$= 4R\frac{(s-a)(s-b)(s-c)}{abc}$

$= 4\left(\frac{abc}{4\Delta}\right)\frac{(s-a)(s-b)(s-c)}{abc}$ $\because R = \frac{abc}{4\Delta}$

$= \frac{(s-a)(s-b)(s-c)}{\Delta}$

$= \frac{s(s-a)(s-b)(s-c)}{s\Delta}$

$= \frac{\Delta^2}{s\Delta}$ $\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta^2 = s(s-a)(s-b)(s-c)$

$= \frac{\Delta}{s}$

$= r = \text{L.H.S}$

Solution

(ii) $\text{R.H.S} = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$= 4R\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$

$= 4R\sqrt{\frac{s^2\cdot s(s-a)(s-b)(s-c)}{(bc)(ac)(ab)}}$

$= 4R\sqrt{\frac{s^2\Delta^2}{a^2b^2c^2}}$ $\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta^2 = s(s-a)(s-b)(s-c)$

$= 4R\frac{s\Delta}{abc}$

$= 4\left(\frac{abc}{4\Delta}\right)s\frac{\Delta}{abc}$ $\because R = \frac{abc}{4\Delta}$

$= s = \text{L.H.S}$

(ii)

(iii) $\text{R.H.S} = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$= 4Rr\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{s(s-c)}{ab}}$$

$$= 4Rr\sqrt{\frac{s(s-a)\cdot s(s-b)\cdot s(s-c)}{(bc)(ac)(ab)}}$$

$$= 4Rr\sqrt{\frac{s^2\cdot s(s-a)(s-b)(s-c)}{a^2b^2c^2}}$$

$$= 4Rr\sqrt{\frac{s^2\cdot\Delta^2}{a^2b^2c^2}}$$

$$= 4Rr\,\frac{s\Delta}{abc}$$

$$= 4\left(\frac{abc}{4\Delta}\right)\left(\frac{\Delta}{s}\right)\frac{s\Delta}{abc}$$

$$= \Delta = \text{L.H.S}$$

$\because \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$

$\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta^2 = s(s-a)(s-b)(s-c)$

$\because R = \frac{abc}{4\Delta} \qquad r = \frac{\Delta}{s}$

Question # 9 Show that

(i) $\dfrac{1}{2rR} = \dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca}$ (ii) $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$

Solution

(i) L.H.S $= \dfrac{1}{2rR}$

$= \dfrac{1}{2\left(\dfrac{\Delta}{s}\right)\left(\dfrac{abc}{4\Delta}\right)}$ $\because r = \dfrac{\Delta}{s}$ $R = \dfrac{abc}{4\Delta}$

$= \dfrac{4s\Delta}{2\Delta abc}$

$= \dfrac{2s}{abc}$

$= \dfrac{a+b+c}{abc}$ $\because 2s = a+b+c$

$= \dfrac{a}{abc} + \dfrac{b}{abc} + \dfrac{c}{abc}$

$= \dfrac{1}{bc} + \dfrac{1}{ac} + \dfrac{1}{ab}$

$= \dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca} =$ R.H.S

(ii) R.H.S $= \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$ $\because r_1 = \dfrac{\Delta}{s-a}$

$= \dfrac{1}{\dfrac{\Delta}{s-a}} + \dfrac{1}{\dfrac{\Delta}{s-b}} + \dfrac{1}{\dfrac{\Delta}{s-c}}$ $r_2 = \dfrac{\Delta}{s-b}$

$r_3 = \dfrac{\Delta}{s-c}$

$= \dfrac{s-a}{\Delta} + \dfrac{s-b}{\Delta} + \dfrac{s-c}{\Delta}$

$= \dfrac{s-a+s-b+s-c}{\Delta}$

$= \dfrac{3s-(a+b+c)}{\Delta}$

$= \dfrac{3s-2s}{\Delta}$ $\because 2s = a+b+c$

$= \dfrac{s}{\Delta}$

$= \dfrac{1}{\Delta/s}$

$= \dfrac{1}{r}$ $\because r = \dfrac{\Delta}{s}$

$=$ L.H.S

Question # 10 Prove that:

(i) $r = \dfrac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$ (ii) $r = \dfrac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$

Solution

(i) R.H.S $= \dfrac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\frac{1}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{bc}{s(s-a)}}$$

$$= a\sqrt{\frac{(s-a)(s-c)(s-a)(s-b)(bc)}{(ac)(ab)s(s-a)}}$$

$$= a\sqrt{\frac{(s-a)(s-b)(s-c)}{a^2 s}} \quad = a\sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2 s^2}}$$

$$= a\frac{\sqrt{s(s-a)(s-b)(s-c)}}{as} \quad = \frac{\Delta}{s} \quad = r = \text{L.H.S}$$

(ii) R.H.S $= \dfrac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$

$\downarrow$

Question # 11 Prove that: $abc(\sin\alpha+\sin\beta+\sin\gamma)=4\Delta s$

Solution

$$\text{L.H.S} = abc(\sin\alpha+\sin\beta+\sin\gamma)$$

$$= abc\left(\frac{2\Delta}{bc}+\frac{2\Delta}{ac}+\frac{2\Delta}{ab}\right)$$

$\because\ \Delta=\frac{1}{2}ab\sin\gamma=\frac{1}{2}bc\sin\alpha=\frac{1}{2}ca\sin\beta$

$\Rightarrow \sin\gamma=\frac{2\Delta}{ab},\quad \sin\alpha=\frac{2\Delta}{bc},\quad \sin\beta=\frac{2\Delta}{ca}$

$$= abc\left(\frac{2\Delta a+2\Delta b+2\Delta c}{abc}\right)$$

$$= 2\Delta a+2\Delta b+2\Delta c$$

$$= 2\Delta(a+b+c)$$

$$= 2\Delta(2s)=4\Delta s = \text{R.H.S} \qquad \because\ 2s=a+b+c$$

Question # 12 Prove that: (i) $(r_1+r_2)\tan\frac{\gamma}{2}=c$ (ii) $(r_3-r)\cot\frac{\gamma}{2}=c$

Solution

(i) $\text{L.H.S} = (r_1+r_2)\tan\frac{\gamma}{2}$

$$= \left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \qquad \because\ \tan\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \left(\frac{\Delta(s-b)+\Delta(s-a)}{(s-a)(s-b)}\right)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}\cdot\frac{s(s-c)}{s(s-c)}}$$

$$= \Delta\left(\frac{s-b+s-a}{(s-a)(s-b)}\right)\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-c)^2}}$$

$$= \Delta\left(\frac{2s-a-b}{(s-a)(s-b)}\right)\sqrt{\frac{\Delta^2}{s^2(s-c)^2}} \qquad \because\ 2s=a+b+c$$

$$= \Delta\left(\frac{a+b+c-a-b}{(s-a)(s-b)}\right)\frac{\Delta}{s(s-c)}$$

$$= \frac{\Delta^2 c}{s(s-a)(s-b)(s-c)} = \frac{\Delta^2 c}{\Delta^2} = c = \text{R.H.S} \qquad \because\ \Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

(ii) $\text{L.H.S} = (r_3-r)\cot\frac{\gamma}{2}$

$$= \left(\frac{\Delta}{s-c}-\frac{\Delta}{s}\right)\frac{1}{\tan\frac{\gamma}{2}} = \Delta\left(\frac{1}{s-c}-\frac{1}{s}\right)\frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$$

$$= \Delta\left(\frac{s-(s-c)}{s(s-c)}\right)\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \qquad \because\ \tan\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \Delta\left(\frac{c}{s(s-c)}\right)\sqrt{\frac{s(s-c)}{(s-a)(s-b)}\cdot\frac{s(s-c)}{s(s-c)}}$$

$$= \Delta\left(\frac{c}{s(s-c)}\right)\frac{s\ (s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \Delta\left(\frac{c}{s(s-c)}\right)\frac{s(s-c)}{\Delta} = c = \text{R.H.S} \qquad \because\ \Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

Question # 2 Show that:

$$r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

Solution

take $a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$

$$= a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\frac{1}{\cos\frac{\alpha}{2}}$$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\frac{1}{\sqrt{\frac{s(s-a)}{bc}}}$$

$\because \sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{bc}{s(s-a)}}$$

$\because \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$$= a\sqrt{\frac{(s-a)(s-c)(s-a)(s-b)(bc)}{(ac)(ab)s(s-a)}}$$

$$= a\sqrt{\frac{(s-a)(s-b)(s-c)}{a^2s}} \qquad = a\sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2s^2}}$$

$$= a\frac{\sqrt{s(s-a)(s-b)(s-c)}}{as} \qquad = \frac{\Delta}{s} = r$$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\Rightarrow a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = r \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Similarly prove yourself

$$b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2} = r \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$$c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2} = r \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

$\downarrow$

From (i), (ii) and (iii)

$$r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

Question # 3 Show that:

(i) $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$ (ii) $r_2 = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$ (iii) $r_3 = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

Solution

(i) R.H.S $= 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$= 4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= 4R\sqrt{\frac{(s-b)(s-c)s(s-b)s(s-c)}{(bc)(ac)(ab)}}$$

$$= 4R\sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= 4R\frac{s(s-b)(s-c)}{abc}$$

$$= 4\frac{abc}{4\Delta}\frac{s(s-b)(s-c)}{abc}\cdot\frac{(s-a)}{(s-a)} \qquad \because\ R = \frac{abc}{4\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)}$$

$$= \frac{\Delta^2}{\Delta(s-a)}$$

$$= \frac{\Delta}{(s-a)}$$

$$= r_1 = \text{R.H.S}$$

(ii) & (iii) $\downarrow$

Question # 4 Show that:

(i) $r_1 = s\tan\frac{\alpha}{2}$ (ii) $r_2 = s\tan\frac{\beta}{2}$ (iii) $r_3 = s\tan\frac{\gamma}{2}$

Solution

(i) R.H.S $= s\tan\frac{\alpha}{2}$

$$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$\because \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}\cdot\frac{s(s-a)}{s(s-a)}}$$

$$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}}$$

$$= s\sqrt{\frac{\Delta^2}{s^2(s-a)^2}}$$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta^2 = s(s-a)(s-b)(s-c)$

$$= s\,\frac{\Delta}{s(s-a)}$$

$$= \frac{\Delta}{(s-a)}$$

$\because r_1 = \frac{\Delta}{s-a}$

$$= r_1$$

$=$ L.H.S

(ii) & (iii)

$\downarrow$

Question # 5 Prove that:

(i) $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$ (ii) $r r_1 r_2 r_3 = \Delta^2$

(iii) $r_1 + r_2 + r_3 - r = 4R$ (iv) $r_1 r_2 r_3 = r s^2$

Solution

(i) L.H.S $= r_1 r_2 + r_2 r_3 + r_3 r_1$

$\because r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

$$= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right) + \left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right) + \left(\frac{\Delta}{s-c}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)}$$

$$= \Delta^2\left(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\right)$$

$$= \Delta^2\left(\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}\right)$$

$$= \Delta^2\left(\frac{3s-(a+b+c)}{(s-a)(s-b)(s-c)}\right)$$

$$= \Delta^2\left(\frac{3s-2s}{(s-a)(s-b)(s-c)}\right) \qquad \because s = \frac{a+b+c}{2}$$

$$= \Delta^2\left(\frac{s}{(s-a)(s-b)(s-c)} \cdot \frac{s}{s}\right)$$

$$= \Delta^2\left(\frac{s^2}{s(s-a)(s-b)(s-c)}\right)$$

$$= \Delta^2\left(\frac{s^2}{\Delta^2}\right) \qquad \because \Delta = \sqrt{s(s-a)(s-b)(s-c)}, \quad \Delta^2 = s(s-a)(s-b)(s-c)$$

$= s^2$ = R.H.S

(ii) L.H.S $= r r_1 r_2 r_3$

$\because r = \dfrac{\Delta}{s}$, $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

$$= \left(\frac{\Delta}{s}\right)\left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right)$$

$$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^4}{\Delta^2} \qquad \because \Delta = \sqrt{s(s-a)(s-b)(s-c)}, \quad \Delta^2 = s(s-a)(s-b)(s-c)$$

$= \Delta^2$ = R.H.S

(iii) L.H.S $= r_1 + r_2 + r_3 - r$

$$= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s}$$

$$= \Delta\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}\right)$$

$$= \Delta\left(\frac{(s-b)+(s-a)}{(s-a)(s-b)} + + \frac{s-(s-c)}{s(s-c)}\right)$$

$$= \Delta\left(\frac{2s-b-a}{(s-a)(s-b)} + \frac{s-s+c}{s(s-c)}\right)$$

$$= \Delta\left(\frac{a+b+c-b-a}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right) \qquad \because\ 2s = a+b+c$$

$$= \Delta\left(\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right)$$

$$= c\Delta\left(\frac{1}{(s-a)(s-b)} + \frac{1}{s(s-c)}\right)$$

$$= c\Delta\left(\frac{s(s-c)-(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right)$$

$$= c\Delta\left(\frac{s^2 - sc + s^2 - as - bs + ab}{\Delta^2}\right)$$

$$= c\left(\frac{2s^2 - s(a+b+c) + ab}{\Delta}\right)$$

$$= c\left(\frac{2s^2 - s(2s) + ab}{\Delta}\right)$$

$$= c\left(\frac{2s^2 - 2s^2 + ab}{\Delta}\right)$$

$$= \frac{abc}{\Delta} = 4\cdot\frac{abc}{4\Delta} = 4R = \text{R.H.S} \qquad \because\ R = \frac{abc}{4\Delta}$$

(iv) L.H.S $= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right)$

$$= \frac{\Delta^3}{(s-a)(s-b)(s-c)}$$

$$= \frac{s\Delta^3}{s(s-a)(s-b)(s-c)}$$

$$= \frac{s\Delta^3}{\Delta^2} = s\Delta$$

$$= s^2\frac{\Delta}{s} = s^2 r = rs^2 = \text{R.H.S}$$

Question # 6 Find R, r, r_1, r_2 and r_3, if measures of the sides of triangle ABC are

(i) $a=13$, $b=14$, $c=15$ (ii) $a=34,$ $b=20,$, $c=42$

Solution

(i) $a=13$, $b=14$, $c=15$

$$s=\frac{a+b+c}{2}=\frac{13+14+15}{2}=21$$

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$
$$=\sqrt{21(21-13)(21-14)(21-15)}$$
$$=\sqrt{21(8)(7)(6)} =\sqrt{7056} =84$$

Now $R=\frac{abc}{4\Delta}=\frac{(13)(14)(15)}{4(84)}=8.125$

$$r=\frac{\Delta}{s} =\frac{84}{21}=4$$

$$r_1=\frac{\Delta}{s-a}=\frac{84}{8}=10.5$$

$$r_2=\frac{\Delta}{s-b}=\frac{84}{7}=12$$

$$r_3=\frac{\Delta}{s-c}=\frac{84}{6}=14$$

(ii) $a=34,$ $b=20,$, $c=42$

$\downarrow$

Question # 7 Prove that in an equilateral triangle,

(i) $r:R:r_1 = 1:2:3$ (ii) $r:R:r_1:r_2:r_3 = 1:2:3:3:3$

Solution

(ii) In equilateral triangle all the sides are equal so $a=b=c$

$$s = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$s-a = \frac{3a}{2} - a = \left(\frac{3}{2}-1\right)a = \frac{1}{2}a$$

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{s(s-a)(s-a)(s-a)}$$

$$= \sqrt{s(s-a)^3}$$

$$= \sqrt{\frac{3a}{2}\left(\frac{1}{2}a\right)^3}$$

$$= \sqrt{\frac{3a}{2}\left(\frac{a^3}{8}\right)} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}\,a^2}{4}$$

Now $r = \dfrac{\Delta}{s}$

$$= \frac{\sqrt{3}\,a^2/4}{3a/2} = \frac{\sqrt{3}\,a^2}{4}\cdot\frac{2}{3a} = \frac{\sqrt{3}\,a}{6}$$

Now

$$r:R:r_1:r_2:r_3 = \frac{\sqrt{3}\,a}{6} : \frac{\sqrt{3}\,a}{3} : \frac{\sqrt{3}\,a}{2} : \frac{\sqrt{3}\,a}{2} : \frac{\sqrt{3}\,a}{2}$$

$$= 1 : 2 : 3 : 3 : 3 \qquad \times\text{ing by } \frac{6}{\sqrt{3}\,a}$$

(i) *Do yourself*

$\downarrow$

Question # 8 Prove that:

(i) $\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$ (ii) $r = s\tan\frac{\alpha}{2} \tan\frac{\beta}{2} \tan\frac{\gamma}{2}$

(iii) $\Delta = 4Rr \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

Solution

(i) R.H.S $= r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

$$= r^2 \frac{1}{\tan\frac{\alpha}{2}} \cdot \frac{1}{\tan\frac{\beta}{2}} \cdot \frac{1}{\tan\frac{\gamma}{2}}$$

$$= r^2 \frac{1}{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}} \cdot \frac{1}{\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}} \cdot \frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$$

$$= r^2 \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= r^2 \sqrt{\frac{s^3(s-a)(s-b)(s-c)}{(s-a)^2(s-b)^2(s-c)^2}}$$

$$= r^2 \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}}$$

$$= r^2 \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)} \cdot \frac{s}{s}}$$

$$= r^2 \sqrt{\frac{s^4}{s(s-a)(s-b)(s-c)}}$$

$$= r^2 \sqrt{\frac{s^4}{\Delta^2}}$$

$$= r^2 \frac{s^2}{\Delta}$$

$$= \left(\frac{\Delta}{s}\right)^2 \frac{s^2}{\Delta} \qquad \because\ r = \frac{\Delta}{s}$$

$$= \frac{\Delta^2}{s^2} \frac{s^2}{\Delta}$$

$= \Delta$ = L.H.S