

INTEGRATION

Exercise 3.4 (Solution) for Class XII

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Question # 1(i) $\int x \sin x dx$

Solution

$$\begin{aligned} & \int x \sin x dx \\ & \text{Integration by parts} \\ & = x \cdot (-\cos x) - \int (-\cos x) \cdot (1) dx \quad \left| \begin{array}{l} u = x \\ v = \sin x \end{array} \right. \\ & = -x \cos x + \int \cos x dx \\ & = -x \cos x + \sin x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(ii) $\int \ln x dx$

Solution

$$\begin{aligned} & \int \ln x dx \\ & = \int \ln x \cdot 1 dx \\ & \text{Integrating by parts} \\ & = \ln x \cdot x - \int x \cdot \frac{1}{x} dx \quad \left| \begin{array}{l} u = \ln x \\ v = 1 \end{array} \right. \\ & = x \ln x - \int dx \\ & = x \ln x - x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(iii) $\int x \ln x dx$

Solution

$$\begin{aligned} & \int x \ln x dx \\ & \text{Integrating by parts} \\ & = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \quad \left| \begin{array}{l} u = \ln x \\ v = x \end{array} \right. \\ & = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx \\ & = \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + c \\ & = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(iv) $\int x^2 \ln x dx$

Solution

$$\begin{aligned} & \int x^2 \ln x dx \\ & \text{Integrating by parts} \\ & = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx \quad \left| \begin{array}{l} u = \ln x \\ v = x^2 \end{array} \right. \\ & = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx \\ & = \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + c \\ & = \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(v) $\int x^3 \ln x dx$

Now do yourself

Solution

$$\begin{aligned} & \int x^3 \ln x dx \\ & \text{Integrating by parts} \\ & = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx \\ & = \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx \\ & = \frac{x^4}{4} \ln x - \frac{1}{4} \cdot \frac{x^4}{4} \\ & = \frac{x^4}{4} \left(\ln x - \frac{1}{4} \right) + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{array}{l|l} u = \ln x \\ v = x^3 \end{array}$$

Question # 1(vi) $\int x^4 \ln x dx$ **Solution**

$$\begin{aligned} & \int x^4 \ln x dx \\ & \text{Integrating by parts} \\ & = \ln x \cdot \frac{x^5}{5} - \int \frac{x^5}{5} \cdot \frac{1}{x} dx \\ & = \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 dx \\ & = \frac{x^5}{5} \ln x - \frac{1}{5} \cdot \frac{x^5}{5} + c \\ & = \frac{x^5}{5} \left(\ln x - \frac{1}{5} \right) + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{array}{l|l} u = \ln x \\ v = x^4 \end{array}$$

Question # 1(vii) $\int \tan^{-1} x dx$ **Solution**

$$\begin{aligned} & \int \tan^{-1} x dx \\ & = \int \tan^{-1} x \cdot 1 dx \\ & \text{Integrating by parts} \\ & = \tan^{-1} x \cdot x - \int x \cdot \frac{1}{1+x^2} dx \\ & = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx \\ & = x \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{array}{l|l} u = \tan^{-1} x \\ v = 1 \end{array}$$

Question # 1(viii) $\int x^2 \sin x dx$ **Solution**

$$\begin{aligned} & \int x^2 \sin x dx \\ & \text{Integrating by parts} \\ & = x^2 (-\cos x) - \int (-\cos x) \cdot 2x dx \\ & = -x^2 \cos x + 2 \int x \cos x dx \end{aligned}$$

$$\begin{array}{l|l} u = x^2 \\ v = \sin x \end{array}$$

Again integrating by parts

$$\begin{aligned} I &= -x^2 \cos x + 2 \left(x \sin x - \int \sin x (1) dx \right) \\ &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(ix) $\int x^2 \tan^{-1} x dx$ **Solution**

$$\begin{aligned} & \int x^2 \tan^{-1} x dx \\ & \text{Integrating by parts} \quad \left| \begin{array}{l} u = \tan^{-1} x \\ v = x^2 \end{array} \right. \\ &= \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x dx + -\frac{1}{3} \int \frac{x}{1+x^2} dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{x^2}{2} + -\frac{1}{3} \cdot \frac{1}{2} \int \frac{2x}{1+x^2} dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + -\frac{1}{6} \ln |1+x^2| + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(x) $\int x \tan^{-1} x dx$ **Solution**

$$\begin{aligned} & \int x \tan^{-1} x dx \quad \left| \begin{array}{l} u = \tan^{-1} x \\ v = x \end{array} \right. \\ & \text{Integrating by parts} \\ &= \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \right) dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{1+x^2} dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(xi) $\int x^3 \tan^{-1} x dx$ **Solution**

$$\begin{aligned} & \int x^3 \tan^{-1} x dx \quad \left| \begin{array}{l} u = \tan^{-1} x \\ v = x^3 \end{array} \right. \\ & \text{Integrating by parts} \\ I &= \tan^{-1} x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4}{1+x^2} dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int x^2 dx + \frac{1}{4} \int dx - \frac{1}{4} \int \frac{1}{1+x^2} dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \cdot \frac{x^3}{3} + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{x^3}{12} + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{array}{r} x^2 - 1 \\ 1 + x^2 \overline{)x^4} \\ \underline{-x^4 - x^2} \\ \hline -x^2 \\ \underline{-1 - x^2} \\ \hline 1 \end{array}$$

Question # 1(xii) $\int x^3 \cos x dx$ **Solution**

$$\begin{aligned} & \int x^3 \cos x dx \\ & \text{Integrating by parts} \quad \left| \begin{array}{l} u = x^3 \\ v = \cos x \end{array} \right. \\ & = x^3 (\sin x) - \int (\sin x) \cdot 3x^2 dx \\ & = x^3 \sin x - 3 \int x^2 \sin x dx \\ & \quad \text{Again integrating by parts} \\ & = x^3 \sin x - 3 \left(x^2 (-\cos x) - \int (-\cos x) \cdot 2x dx \right) \\ & = x^3 \sin x + 3x^2 \cos x - 6 \int x \cos x dx \\ & \quad \text{Again integrating by parts} \quad \left| \begin{array}{l} u = x \\ v = \cos x \end{array} \right. \\ & = x^3 \sin x + 3x^2 \cos x - 6 \left(x(\sin x) - \int (\sin x) \cdot 1 dx \right) \\ & = x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6 \int \sin x dx \\ & = x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6 \cos x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(xiii) $\int \sin^{-1} x dx$ **Solution**

$$\begin{aligned} & \int \sin^{-1} x dx \quad \left| \begin{array}{l} u = \sin^{-1} x \\ v = 1 \end{array} \right. \\ & = \int \sin^{-1} x \cdot 1 dx \\ & \quad \text{Integrating by parts} \\ & = \sin^{-1} x \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}} dx \\ & = x \sin^{-1} x - \int (1-x^2)^{-\frac{1}{2}} (x) dx \\ & = x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx \\ & = x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \\ & = x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ & = x \sin^{-1} x + \sqrt{1-x^2} + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(xiv) $\int x \sin^{-1} x dx$ **Solution**

$$\begin{aligned} & \int x \sin^{-1} x dx \quad \left| \begin{array}{l} u = \sin^{-1} x \\ v = x \end{array} \right. \\ & \quad \text{Integrating by parts} \\ & = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx \\ & = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx \\ & = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx \\ & = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right) dx \\ & = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx \\ & = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx \\ & = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} I_1 - \frac{1}{2} \sin^{-1} x \quad \dots \dots (i) \end{aligned}$$

Let $I_1 = \int \sqrt{1-x^2} dx$

Where $I_1 = \int \sqrt{1-x^2} dx$

$$\begin{aligned} &\text{Put } x = \sin \theta \\ &\Rightarrow dx = \cos \theta d\theta \end{aligned}$$

$$\begin{aligned} &= \int \sqrt{1-\sin^2 \theta} \cos \theta d\theta \\ &= \int \sqrt{\cos^2 \theta} \cos \theta d\theta \\ &= \int \cos^2 \theta d\theta = \int \left(\frac{1+\cos 2\theta}{2} \right) d\theta \\ &= \frac{1}{2} \int (1+\cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] + c \\ &= \frac{1}{2} \left[\theta + \frac{2\sin \theta \cos \theta}{2} \right] + c \\ &= \frac{1}{2} \left[\theta + \sin \theta \sqrt{1-\sin^2 \theta} \right] + c \\ &= \frac{1}{2} \left[\sin^{-1} x + x\sqrt{1-x^2} \right] + c \end{aligned}$$

Using value of I_1 in (i)

$$\begin{aligned} I &= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{1}{2} (\sin^{-1} x + x\sqrt{1-x^2}) + c \right] - \frac{1}{2} \sin^{-1} x \\ &= \frac{x^2}{2} \sin^{-1} x + \frac{1}{4} \sin^{-1} x + \frac{1}{4} x\sqrt{1-x^2} + \frac{1}{2} c - \frac{1}{2} \sin^{-1} x \\ I &= \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x\sqrt{1-x^2} + \frac{1}{2} c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(xv) $\int e^x \sin x \cos x dx$

Solution

Let $I = \int e^x \sin x \cos x dx$

$$= \frac{1}{2} \int e^x \cdot 2 \sin x \cos x dx$$

$$= \frac{1}{2} \int e^x \sin 2x dx$$

Integrating by parts

$$= \frac{1}{2} \left[e^x \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot e^x dx \right]$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \int e^x \cos 2x dx$$

Again integrating by parts

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} e^x \right)$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - \frac{1}{2} \int e^x \sin 2x \right)$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - I \right) + c$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x - \frac{1}{4} I + c$$

$$I + \frac{1}{4} I = -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c$$

$$\frac{5}{4} I = -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c$$

$$I = -\frac{1}{5} e^x \cos 2x + \frac{1}{10} e^x \sin 2x + \frac{4}{5} c \quad \underline{\text{Ans.}}$$

Question # 1(xvi) $\int x \sin x \cos x dx$ **Solution**

$$\text{Let } I = \int x \sin x \cos x dx$$

$$\begin{aligned} &= \frac{1}{2} \int x \cdot 2 \sin x \cos x dx \\ &= \frac{1}{2} \int x \cdot \sin 2x dx \quad \left| \begin{array}{l} u = x \\ v = \sin 2x \end{array} \right. \\ &\text{Integrating by parts} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left[x \left(-\frac{\cos 2x}{2} \right) - \int \left(-\frac{\cos 2x}{2} \right) (1) dx \right] \\ &= -\frac{x \cos 2x}{4} + \frac{1}{4} \int \cos 2x dx \\ &= -\frac{x \cos 2x}{4} + \frac{1}{4} \left(\frac{\sin 2x}{2} \right) + c \\ &= -\frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(xvii) $\int x \cos^2 x dx$ **Solution**

$$\text{Let } I = \int x \cos^2 x dx$$

$$\begin{aligned} &= \int x \left(\frac{1 + \cos 2x}{2} \right) dx \\ &= \frac{1}{2} \int x (1 + \cos 2x) dx \quad \left| \begin{array}{l} u = x \\ v = \cos 2x \end{array} \right. \\ &= \frac{1}{2} \int x dx + \frac{1}{2} \int x \cos 2x dx \\ &= \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2} \left[x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot (1) dx \right] \\ &= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x - \frac{1}{4} \int \sin 2x dx \\ &= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x - \frac{1}{4} \left(-\frac{\cos 2x}{2} \right) + c \\ &= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x + \frac{1}{8} \cos 2x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(xviii) $\int x \sin^2 x dx$ **Solution**

$$\begin{aligned} &= \int x \sin^2 x dx \\ &= \int x \left(\frac{1 - \cos 2x}{2} \right) dx = \frac{1}{2} \int x (1 - \cos 2x) dx \\ &= \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx \quad \left| \begin{array}{l} u = x \\ v = \cos 2x \end{array} \right. \\ &\text{Integrating by parts} \\ &= \frac{1}{2} \cdot \frac{x^2}{2} - \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot (1) dx \right] \\ &= \frac{x^2}{4} - \frac{1}{4} x \sin 2x + \frac{1}{4} \int \sin 2x dx \\ &= \frac{x^2}{4} - \frac{1}{4} x \sin 2x + \frac{1}{4} \left(-\frac{\cos 2x}{2} \right) + c \\ &= \frac{x^2}{4} - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 1(ix) $\int (\ln x)^2 dx$ **Solution**

$$\begin{aligned} \text{Let } I &= \int (\ln x)^2 dx \\ &= \int (\ln x)^2 \cdot 1 dx \end{aligned} \quad \left| \begin{array}{l} u = (\ln x)^2 \\ v = 1 \end{array} \right.$$

Integrating by parts

$$= (\ln x)^2 \cdot x - \int x \cdot 2(\ln x) \cdot \frac{1}{x} dx$$

$$= x(\ln x)^2 - 2 \int (\ln x) dx$$

Again integrating by parts

$$= x(\ln x)^2 - 2 \left[\ln x \cdot x - \int x \cdot \frac{1}{x} dx \right]$$

$$= x(\ln x)^2 - 2x \ln x + 2 \int dx$$

$$= x(\ln x)^2 - 2x \ln x + 2x + c \quad \underline{\text{Ans.}}$$

Question # 1(xx) $\int \ln(\tan x) \sec^2 x dx$ **Solution**

$$\text{Let } I = \int \ln(\tan x) \sec^2 x dx \quad \left| \begin{array}{l} u = \ln(\tan x) \\ v = \sec^2 x \end{array} \right.$$

Integrating by parts

$$I = \ln(\tan x) \cdot \tan x - \int \tan x \cdot \frac{1}{\tan x} \cdot \sec^2 x dx$$

$$= \tan x \ln(\tan x) - \int \sec^2 x dx$$

$$= \tan x \ln(\tan x) - \tan x + c \quad \underline{\text{Ans.}}$$

Question # 1(xxi) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ **Solution**

$$\begin{aligned} \text{Let } I &= \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx \\ &= \int \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} (x) dx \quad \left| \begin{array}{l} u = \sin^{-1} x \\ v = (1-x^2)^{-\frac{1}{2}} (-2x) \end{array} \right. \\ &= -\frac{1}{2} \int \sin^{-1} x \cdot (1-x^2)^{-\frac{1}{2}} (-2x) dx \end{aligned}$$

Integrating by parts

$$I = -\frac{1}{2} \left[\sin^{-1} x \cdot \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \cdot \frac{1}{\sqrt{1-x^2}} dx \right]$$

$$= -\frac{1}{2} \left[\sin^{-1} x \cdot \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \int \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} \cdot \frac{1}{\sqrt{1-x^2}} dx \right]$$

$$= -\frac{1}{2} \left[2(1-x^2)^{\frac{1}{2}} \sin^{-1} x - 2 \int dx \right]$$

$$= -\frac{1}{2} \left[(1-x^2)^{\frac{1}{2}} \sin^{-1} x - \int dx \right]$$

$$= -\sqrt{1-x^2} \sin^{-1} x + \int dx$$

$$= -\sqrt{1-x^2} \sin^{-1} x + x + c$$

$$= x - \sqrt{1-x^2} \sin^{-1} x + c \quad \underline{\text{Ans.}}$$

Question # 2 (i) $\int \tan^4 x dx$ **Solution**

$$\begin{aligned}
\text{Let } I &= \int \tan^4 x dx \\
&= \int \tan^2 x \cdot \tan^2 x dx \\
&= \int \tan^2 x (\sec^2 x - 1) dx \\
&= \int (\tan^2 x \sec^2 x - \tan^2 x) dx \\
&= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx \\
&= \int \tan^2 x \frac{d}{dx}(\tan x) dx - \int (\sec^2 x - 1) dx \\
&= \frac{\tan^{2+1} x}{2+1} - \int \sec^2 x dx - \int dx \\
&= \frac{1}{3} \tan^3 x - \tan x - x + c \quad \underline{\text{Ans.}}
\end{aligned}$$

Question # 2(ii) $\int \sec^4 x dx$ **Solution**

$$\begin{aligned}
\text{Let } I &= \int \sec^4 x dx \\
&= \int (\sec^2 x) \cdot (\sec^2 x) dx \\
&= \int (1 + \tan^2 x) \cdot (\sec^2 x) dx \\
&= \int \sec^2 x dx + \int \tan^2 x \sec^2 x dx \\
&= \tan x + \int (\tan x)^2 \frac{d}{dx}(\tan x) dx \\
&= \tan x + \frac{\tan^3 x}{3} + c \quad \underline{\text{Ans.}}
\end{aligned}$$

Question # 2(iii) $\int e^x \sin 2x \cos x dx$ **Solution**

$$\begin{aligned}
\text{Let } I &= \int e^x \sin 2x \cos x dx \\
&= \frac{1}{2} \int e^x (2 \sin 2x \cos x) dx \\
&= \frac{1}{2} \int e^x (\sin(2x+x) + \sin(2x-x)) dx \\
&= \frac{1}{2} \int e^x (\sin 3x + \sin x) dx \\
&= \frac{1}{2} \int e^x \sin 3x dx + \frac{1}{2} \int e^x \sin x dx \\
&= \frac{1}{2} I_1 + \frac{1}{2} I_2 \quad \dots \dots \dots (i)
\end{aligned}$$

Where $I_1 = \int e^x \sin 3x dx$ and $I_2 = \int e^x \sin x dx$ Solve I_1 and I_2 as in Q # 1(xv) and put value of I_1 and I_2 in (i).**Question # 1(xv)**

Let $I = \int e^x \sin x \cos x dx \quad \left| \begin{array}{l} u = e^x \\ v = \sin 2x \end{array} \right.$

$= \frac{1}{2} \int e^x \cdot 2 \sin x \cos x dx$

$= \frac{1}{2} \int e^x \sin 2x dx \quad \because \sin 2x = 2 \sin x \cos x$

Integrating by parts

$I = \frac{1}{2} \left[e^x \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot e^x dx \right]$

$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \int e^x \cos 2x dx$

Again integrating by parts

$I = -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} e^x \right)$

$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - \frac{1}{2} \int e^x \sin 2x \right)$

$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4} \left(e^x \cdot \frac{\sin 2x}{2} - I \right) + c$

$= -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x - \frac{1}{4} I + c$

$\Rightarrow I + \frac{1}{4} I = -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c$

$\Rightarrow \frac{5}{4} I = -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c$

$\Rightarrow I = -\frac{1}{5} e^x \cos 2x + \frac{1}{10} e^x \sin 2x + \frac{4}{5} c$

Question # 2(iv) $\int \tan^3 x \cdot \sec x dx$ **Solution**

$$\begin{aligned}
I &= \int \tan^3 x \cdot \sec x dx \\
&= \int \tan^2 x \cdot \tan x \cdot \sec x dx \\
&= \int (\sec^2 x - 1) \cdot \sec x \tan x dx \quad \text{Put } t = \sec x \\
&\qquad\qquad\qquad \Rightarrow dt = \sec x \tan x dx \\
&= \int (t^2 - 1) dt \\
&= \frac{t^3}{3} - t + c \\
&= \frac{\sec^3 x}{3} - \sec x + c \quad \underline{\text{Ans.}}
\end{aligned}$$

Question # 2(v) $\int x^3 e^{5x} dx$ **Solution**

Let $I = \int x^3 e^{5x} dx$ | $u = x^3$
 Integrating by parts | $v = e^x$

$$\begin{aligned}
I &= x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 3x^2 dx \\
&= \frac{1}{5} x^3 e^{5x} - \frac{3}{5} \int x^2 e^{5x} dx \quad | \quad u = x^2 \\
&\qquad\qquad\qquad v = e^{5x}
\end{aligned}$$

Again integrating by parts

$$\begin{aligned}
I &= \frac{1}{5} x^3 e^{5x} - \frac{3}{5} \left[x^2 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 2x dx \right] \\
&= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \int x e^{5x} dx
\end{aligned}$$

Again integrating by parts

$$\begin{aligned}
&= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \left[x \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot (1) dx \right] \\
&= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125} \int e^{5x} dx \\
&= \frac{1}{5} x^3 e^{5x} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125} \cdot \frac{e^{5x}}{5} + c \\
&= \frac{e^{5x}}{5} \left(x^3 - \frac{3}{5} x^2 + \frac{6}{25} x - \frac{6}{125} \right) + c \quad \underline{\text{Ans.}}
\end{aligned}$$

Question 2(vi) $\int e^{-x} \sin 2x dx$ **Solution**

Let $I = \int e^{-x} \sin 2x dx$ | $u = e^{-x}$
 Integrating by parts | $v = \sin 2x$

$$\begin{aligned}
&= e^{-x} \cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2} \cdot e^{-x} (-1) dx \\
&= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{2} \int e^{-x} \cos 2x dx
\end{aligned}$$

Again integrating by parts

$$\begin{aligned}
&= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{2} \left[e^{-x} \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot e^{-x} (-1) dx \right] \\
&= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x - \frac{1}{4} \int e^{-x} \sin 2x dx \\
&= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x - \frac{1}{4} I + c
\end{aligned}$$

$$I + \frac{1}{4} I = -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x + c$$

$$\frac{5}{4} I = -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x + c$$

$$\begin{aligned}
I &= -\frac{2}{5} e^{-x} \cos 2x - \frac{1}{5} e^{-x} \sin 2x + \frac{4}{5} c \\
&= -\frac{1}{5} e^{-x} (2 \cos 2x + \sin 2x) + \frac{4}{5} c \quad \underline{\text{Ans.}}
\end{aligned}$$

Question # 2(vii) $\int e^{2x} \cdot \cos 3x dx$ **Solution**

$$\text{Let } I = \int e^{2x} \cdot \cos 3x dx$$

Integrating by parts

$$= e^{2x} \cdot \frac{\sin 3x}{3} - \int \frac{\sin 3x}{3} \cdot e^{2x}(2) dx$$

$$= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx$$

Again integrating by parts

$$= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \left[e^{2x} \cdot \frac{-\cos 3x}{3} - \int \frac{-\cos 3x}{3} \cdot e^{2x}(2) dx \right]$$

$$= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cdot \cos 3x dx$$

$$I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} I + c$$

$$I + \frac{4}{9} I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c$$

$$\frac{13}{9} I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c$$

$$I = \frac{9}{13} \left[\frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c \right]$$

$$I = \left[\frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x + \frac{c}{13} \right] \underline{\text{Ans.}}$$

Question # 2(viii) $\int \operatorname{cosec}^3 x dx$ **Solution**

$$\text{Let } I = \int \operatorname{cosec}^3 x dx$$

$$= \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x dx \quad \left| \begin{array}{l} u = \operatorname{cosec} x \\ v = \operatorname{cosec}^2 x \end{array} \right.$$

Integrating by parts

$$= \operatorname{csc} x (-\cot x) i \int (-\cot x)(-\csc x \cot x) dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) dx$$

$$= -\operatorname{cosec} x \cot x - \int (\operatorname{cosec}^3 x - \operatorname{cosec} x) dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x dx + \int \operatorname{cosec} x dx$$

$$= -\operatorname{cosec} x \cot x - I + \ln |\operatorname{cosec} x - \cot x| + c$$

$$I + I = -\operatorname{cosec} x \cot x + \ln |\operatorname{cosec} x - \cot x| + c$$

$$2I = -\operatorname{cosec} x \cot x + \ln |\operatorname{cosec} x - \cot x| + c$$

$$I = -\frac{1}{2} \operatorname{csc} x \cot x + \frac{1}{2} \ln |\operatorname{csc} x - \cot x| + \frac{1}{2} c \quad \underline{\text{Ans.}}$$

Question # 3 $\int e^{ax} \sin bx dx$ **Solution**

$$\text{Let } I = \int e^{ax} \sin bx dx \quad \left| \begin{array}{l} u = e^{ax} \\ v = \sin bx \end{array} \right.$$

Integrating by parts

$$\begin{aligned} &= e^{ax} \left(-\frac{\cos bx}{b} \right) - \int \left(-\frac{\cos bx}{b} \right) \cdot e^{ax} (a) dx \\ &= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx dx \end{aligned}$$

Again integrating by parts

$$\begin{aligned} &= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \left[e^{ax} \frac{\sin bx}{b} - \int \frac{\sin bx}{b} \cdot e^{ax} a dx \right] \\ &= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx dx \\ &= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I + c_1 \end{aligned}$$

$$\Rightarrow I + \frac{a^2}{b^2} I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx + c_1$$

$$\left(\frac{b^2 + a^2}{b^2} \right) I = \frac{e^{ax}}{b^2} (-b \cos bx + a \sin bx) + c_1$$

$$(b^2 + a^2) I = e^{ax} (a \sin bx - b \cos bx) + b^2 c_1$$

$$\text{Put } a = r \cos \theta \quad \& \quad b = r \sin \theta$$

Squaring and adding

$$\begin{aligned} a^2 + b^2 &= r^2 (\cos^2 \theta + \sin^2 \theta) \\ \Rightarrow a^2 + b^2 &= r^2 (1) \\ \Rightarrow r &= \sqrt{a^2 + b^2} \end{aligned}$$

Also

$$\begin{aligned} \frac{b}{a} &= \frac{r \sin \theta}{r \cos \theta} \Rightarrow \frac{b}{a} = \tan \theta \\ \Rightarrow \theta &= \tan^{-1} \frac{b}{a} \end{aligned}$$

$$(b^2 + a^2) I = e^{ax} (r \cos \theta \sin bx - r \sin \theta \cos bx) + b^2 c_1$$

$$(b^2 + a^2) I = e^{ax} r (\sin bx \cos \theta - \cos bx \sin \theta) + b^2 c_1$$

$$(a^2 + b^2) I = e^{ax} r \sin(bx - \theta) + b^2 c_1$$

Putting value of r and θ

$$(a^2 + b^2) I = e^{ax} \sqrt{a^2 + b^2} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) + b^2 c_1$$

$$\Rightarrow I = \frac{\sqrt{a^2 + b^2}}{(a^2 + b^2)} e^{ax} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) + \frac{b^2}{a^2 + b^2} c_1$$

$$I = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) + c \quad \underline{\text{Ans.}} \quad \text{here } c = \frac{b^2}{a^2 + b^2} c_1$$

Question # 4(i) $\int \sqrt{a^2 - x^2} dx$ **Solution**

$$\text{Let } I = \int \sqrt{a^2 - x^2} dx$$

$$= \int \sqrt{a^2 - x^2} \cdot 1 dx$$

Integrating by parts

$$= \sqrt{a^2 - x^2} \cdot x - \int x \cdot \frac{1}{2} (a^2 - x^2)^{\frac{1}{2}} \cdot (-2x) dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{(a^2 - x^2)^{\frac{1}{2}}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{(a^2 - x^2)^{\frac{1}{2}}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{(a^2 - x^2)^{\frac{1}{2}}} - \frac{a^2}{(a^2 - x^2)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} dx + \int \frac{a^2}{\sqrt{a^2 - x^2}} dx$$

$$I = x\sqrt{a^2 - x^2} - I + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

$$I + I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} + c$$

$$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} + c$$

$$I = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} + \frac{1}{2} c \quad \underline{\text{Ans.}}$$

Question # 4(ii) $\int \sqrt{x^2 - a^2} dx$ **Solution**

$$\text{Let } I = \int \sqrt{x^2 - a^2} dx$$

$$= \int \sqrt{x^2 - a^2} \cdot 1 dx$$

Integrating by parts

$$I = \sqrt{x^2 - a^2} \cdot x - \int x \cdot \frac{1}{2} (x^2 - a^2)^{\frac{1}{2}} \cdot (2x) dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{(x^2 - a^2)^{\frac{1}{2}}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{(x^2 - a^2)^{\frac{1}{2}}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \left(\frac{x^2 - a^2}{(x^2 - a^2)^{\frac{1}{2}}} + \frac{a^2}{(x^2 - a^2)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x\sqrt{x^2 - a^2} - I - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} dx$$

$$I + I = x\sqrt{x^2 - a^2} - a^2 \ln \left| x + \sqrt{x^2 - a^2} \right| + c \quad \therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$

$$2I = x\sqrt{x^2 - a^2} - a^2 \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$

$$I = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + \frac{1}{2} c \quad \underline{\text{Ans.}}$$

Question # 4(iii) $\int \sqrt{4-5x^2} dx$ **Solution**

$$\begin{aligned}
\text{Let } I &= \int \sqrt{4-5x^2} dx \\
&= \int \sqrt{4-5x^2} \cdot 1 dx \\
&\quad \text{Integrating by parts} \\
&= \sqrt{4-5x^2} \cdot x - \int x \cdot \frac{1}{2}(4-5x^2)^{-\frac{1}{2}} \cdot (-10x) dx \\
&= \sqrt{4-5x^2} \cdot x - \int \frac{-5x^2}{(4-5x^2)} dx \\
&= \sqrt{4-5x^2} \cdot x - \int \frac{4-5x^2-4}{(4-5x^2)} dx \\
&= \sqrt{4-5x^2} \cdot x - \int \left(\frac{4-5x^2}{(4-5x^2)^{\frac{1}{2}}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx \\
&= \sqrt{4-5x^2} \cdot x - \int \left((4-5x^2)^{\frac{1}{2}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx \\
&= \sqrt{4-5x^2} \cdot x - \int \sqrt{4-5x^2} dx + 4 \int \frac{1}{\sqrt{4-5x^2}} dx \\
I &= \sqrt{4-5x^2} \cdot x - I + 4 \int \frac{1}{\sqrt{5\left(\frac{4}{5}-x^2\right)}} dx \\
I+I &= \sqrt{4-5x^2} \cdot x + 4 \int \frac{1}{\sqrt{5}\sqrt{\frac{4}{5}-x^2}} dx \\
2I &= \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2-x^2}} dx \\
&= \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} \sin^{-1}\left(\frac{x}{2/\sqrt{5}}\right) + c_1 \quad \because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} \\
I &= \frac{x}{2}\sqrt{4-5x^2} + \frac{4}{2\sqrt{5}} \sin^{-1}\left(\frac{\sqrt{5}x}{2}\right) + \frac{1}{2}c_1 \\
&= \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt{5}} \sin^{-1}\left(\frac{\sqrt{5}x}{2}\right) + c \quad \underline{\text{Ans.}} \quad \text{here } c = \frac{1}{2}c_1
\end{aligned}$$

Question # 4(iv) $\int \sqrt{3-4x^2} dx$ **Solution**

$$\text{Let } I = \int \sqrt{3-4x^2} dx$$

Question # 4(v) $\int \frac{dx}{\sqrt{x^2 + 4}}$

Solution
Do yourself as Question 4(ii)

Question # 4(vi) $\int x^2 e^{ax} dx$

Let $I = \int x^2 e^{ax} dx$

Do yourself as Question # 2(v)

Question # 5(i) $\int e^x \left(\frac{1}{x} + \ln x \right) dx$

Solution

$$\begin{aligned} \text{Let } I &= \int e^x \left(\frac{1}{x} + \ln x \right) dx \\ &= \int e^x \left(\ln x + \frac{1}{x} \right) dx \\ &= \int e^x (f(x) + f'(x)) dx \\ &= e^x f(x) + c = e^x \ln x + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{aligned} \text{Put } f(x) &= \ln x \\ \Rightarrow f'(x) &= \frac{1}{x} \end{aligned}$$

Question # 5(ii) $\int e^x (\cos x + \sin x) dx$

Solution

$$\begin{aligned} \text{Let } I &= \int e^x (\cos x + \sin x) dx \\ &= \int e^x (\sin x + \cos x) dx \\ &= \int e^x (f(x) + f'(x)) dx \\ &= e^x f(x) + c \\ &= e^x \sin x + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{aligned} \text{Put } f(x) &= \sin x \\ \Rightarrow f'(x) &= \cos x \end{aligned}$$

Question # 5(iii) $\int e^{ax} \left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}} \right] dx$

Solution

$$\begin{aligned} \text{Let } I &= \int e^{ax} \left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}} \right] dx \\ &= \int e^{ax} [a f(x) + f'(x)] dx \\ &= e^{ax} f(x) + c \\ &= e^{ax} \sec^{-1} x + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{aligned} \text{Put } f(x) &= \sec^{-1} x \\ \Rightarrow f'(x) &= \frac{1}{x\sqrt{x^2-1}} \end{aligned}$$

Question # 5(iv) $\int e^{3x} \left(\frac{3 \sin x - \cos x}{\sin^2 x} \right) dx$

Solution

$$\begin{aligned} \text{Let } I &= \int e^{3x} \left(\frac{3 \sin x - \cos x}{\sin^2 x} \right) dx \\ &= \int e^{3x} \left(3 \frac{1}{\sin x} - \frac{\cos x}{\sin x \cdot \sin x} \right) dx \\ &= \int e^{3x} (3 \csc x - \csc x \cot x) dx \\ &\quad \text{Put } f(x) = \csc x \\ &\quad \Rightarrow f'(x) = -\csc x \cot x \\ &= \int e^{3x} (3f(x) + f'(x)) dx \\ &= e^{3x} f(x) + c \\ &= e^{3x} \csc x + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{aligned} \text{Put } f(x) &= \csc x \\ \Rightarrow f'(x) &= -\csc x \cot x \end{aligned}$$

Question 5(v) $\int e^{2x} (-\sin x + 2 \cos x) dx$

Solution

$$\begin{aligned} \text{Let } I &= \int e^{2x} (-\sin x + 2 \cos x) dx \\ &= \int e^{2x} (2 \cos x - \sin x) dx \\ &= \int e^{2x} (2f(x) + f'(x)) dx \\ &= e^{2x} f(x) + c \\ &= e^{2x} \cos x + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{aligned} \text{Put } f(x) &= \cos x \\ \Rightarrow f'(x) &= -\sin x \end{aligned}$$

Question # 5(vi) $\int \frac{xe^x}{(1+x)^2} dx$

Solution Let $I = \int \frac{xe^x}{(1+x)^2} dx$

$$\begin{aligned} &= \int \frac{(1+x-1)e^x}{(1+x)^2} dx \\ &= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx \\ &= \int e^x \left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right] dx \\ &= \int e^x (f(x) + f'(x)) dx \\ &= e^x f(x) + c \\ &= e^x \left(\frac{1}{1+x} \right) + c \quad \underline{\text{Ans.}} \end{aligned}$$

Put $f(x) = \frac{1}{1+x} = (1+x)^{-1}$
 $\Rightarrow f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$

Question # 5(vii) $\int e^{-x} (\cos x - \sin x) dx$

Solution

$$\begin{aligned} \text{Let } I &= \int e^{-x} (\cos x - \sin x) dx \\ &= \int e^{-x} ((-1)\sin x + \cos x) dx \\ &= \int e^{-x} ((-1)f(x) + f'(x)) dx \\ &= e^{-x} f(x) + c \\ &= e^{-x} \sin x + c \quad \underline{\text{Ans.}} \end{aligned}$$

Put $f(x) = \sin x$
 $\Rightarrow f'(x) = \cos x$

Question # 5(viii) $\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx \\ &= \int e^{m \tan^{-1} x} \cdot \frac{1}{1+x^2} dx \\ &= \int e^{mt} dt \quad \text{Put } t = \tan^{-1} x \\ &= \frac{e^{mt}}{m} + c \quad \Rightarrow dt = \frac{1}{1+x^2} dx \\ &= \frac{1}{m} e^{m \tan^{-1} x} + c \quad \underline{\text{Ans.}} \end{aligned}$$

Put $t = \tan^{-1} x$
 $\Rightarrow dt = \frac{1}{1+x^2} dx$

Question # 5(ix) $\int \frac{2x}{1-\sin x} dx$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{2x}{1-\sin x} dx \\ &= \int \frac{2x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} dx \\ &= \int \frac{2x(1+\sin x)}{1-\sin^2 x} dx \\ &= \int \frac{2x+2x\sin x}{\cos^2 x} dx \\ &= \int \left(\frac{2x}{\cos^2 x} + \frac{2x\sin x}{\cos^2 x} \right) dx \\ &= 2 \int x \sec^2 x dx + 2 \int x \sec x \tan x dx \end{aligned}$$

Integrating by parts

$$\begin{aligned} &= 2 \left[x \cdot \tan x - \int \tan x \cdot 1 dx \right] + 2 \left[x \cdot \sec x - \int \sec x (1) dx \right] \\ &= 2 \left[x \cdot \tan x - \ln |\sec x| \right] + 2 \left[x \cdot \sec x - \ln |\sec x + \tan x| \right] + c \\ &= 2x \tan x - 2 \ln |\sec x| + 2x \sec x - 2 \ln |\sec x + \tan x| + c \quad \underline{\text{Ans.}} \end{aligned}$$

Question # 5(x) $\int \frac{e^x(1+x)}{(2+x)^2} dx$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{e^x(1+x)}{(2+x)^2} dx \\ &= \int \frac{e^x(2+x-1)}{(2+x)^2} dx \\ &= \int e^x \left(\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right) dx \\ &= \int e^x \left((2+x)^{-1} - (2+x)^{-2} \right) dx \\ &= \int e^x (f(x) + f'(x)) dx \\ &= e^x f(x) + c \\ &= e^x (2+x)^{-1} + c \\ &= \frac{e^x}{2+x} + c \quad \underline{\text{Ans.}} \end{aligned}$$

$$\begin{aligned} \text{Put } f(x) &= (2+x)^{-1} \\ \Rightarrow f'(x) &= -(2+x)^{-2} \end{aligned}$$

Question # 5(xi) $\int \left(\frac{1-\sin x}{1-\cos x} \right) e^x dx$

Solution

$$\begin{aligned} \text{Let } I &= \int \left(\frac{1-\sin x}{1-\cos x} \right) e^x dx \\ &= \int \left(\frac{1-2\sin \frac{x}{2}\cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) e^x dx \\ &= \int \left(\frac{1}{2\sin^2 \frac{x}{2}} - \frac{2\sin \frac{x}{2}\cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) e^x dx \\ &= \int \left(\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right) e^x dx \\ &= \int e^x \left(-\cot \frac{x}{2} + \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right) dx \end{aligned}$$

$$\begin{aligned} \text{Put } f(x) &= -\cot \frac{x}{2} \\ \Rightarrow f'(x) &= \operatorname{cosec}^2 \frac{x}{2} \cdot \frac{1}{2} \\ \Rightarrow f'(x) &= \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \end{aligned}$$

$$\begin{aligned} &= \int e^x (f(x) + f'(x)) \\ &= e^x f(x) + c \\ &= e^x \left(-\cot \frac{x}{2} \right) + c \\ &= -e^x \cot \frac{x}{2} + c \quad \underline{\text{Ans.}} \end{aligned}$$