Question # 1 Express the following products as sums or differences:

(i) $2\sin 3\theta\cos\theta$ (ii) $2\cos 5\theta\cos 3\theta$ (iii) $\sin 5\theta\cos 2\theta$
(iv) $2\sin 7\theta\sin 2\theta$ (v) $\cos(x+y)\sin(x-y)$ (vi) $\cos(2x+30^\circ)\cos(2x-30^\circ)$
(vii) $\sin 12^\circ\sin 46^\circ$ (viii) $\sin(x+45^\circ)\sin(x-45^\circ)$

Solution

(i) Since $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$
$$2\sin 3\theta\cos\theta=\sin(3\theta+\theta)+\sin(3\theta-\theta)$$
$$=\sin 4\theta+\sin 2\theta$$

(ii) Since $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$
$$2\cos 5\theta\cos 3\theta=\cos(5\theta+3\theta)-\cos(5\theta-3\theta)$$
$$=\cos 8\theta-\cos 2\theta$$

(iii) Since $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$
$$2\sin 5\theta\cos 2\theta=\sin(5\theta+2\theta)+\sin(5\theta-2\theta)$$
$$=\sin 7\theta+\sin 3\theta$$
$$\sin 5\theta\cos 2\theta=\frac{\sin 7\theta+\sin 3\theta}{2}$$

(iv) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$
$$-2\sin 7\theta\sin 2\theta=\cos(7\theta+2\theta)-\cos(7\theta-2\theta)$$
$$=\cos 9\theta-\cos 5\theta$$
$$\sin 7\theta\sin 2\theta=\frac{\cos 5\theta-\cos 9\theta}{2}$$

(v) Since $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$
$$2\cos(x+y)\sin(x-y)=\sin(x+y+x-y)-\sin(x+y-x+y)$$
$$=\sin 2x-\sin 2y$$
$$\Rightarrow \cos(x+y)\sin(x-y)=\frac{\sin 2x-\sin 2y}{2}$$

(vi) Since $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$
$$2\cos(2x+30^\circ)\cos(2x-30^\circ)=\cos(2x+30^\circ+2x-30^\circ)$$
$$=\cos(4x)+\cos(60^\circ)+\cos(2x+30^\circ-2x+30^\circ)$$
$$\Rightarrow \quad \cos(2x+30^\circ)\cos(2x-30^\circ)=\frac{\cos 4x+\cos 60^\circ}{2}$$

(vii) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$
$$-2\sin 12^\circ\sin 46^\circ=\cos(12+46)-\cos(12-46) \qquad \because \cos(-\theta)=\cos\theta$$
$$=\cos 58-\cos(-34)$$
$$=\cos 58-\cos 34$$
$$\Rightarrow \quad \sin 12^\circ\sin 46^\circ=\frac{\cos 34-\cos 58}{2}$$

(viii) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$
$$-2\sin(x+45^\circ)\sin(x-45^\circ)=\cos\{(x+45^\circ)+(x-45^\circ)\}-\cos\{(x+45^\circ)-(x-45^\circ)\}$$
$$=\cos 2x-\cos 90^\circ$$
$$\Rightarrow \quad \sin(x+45^\circ)\sin(x-45^\circ)=\frac{\cos 90^\circ-\cos 2x}{2}$$

Question # 2 Express the following sum or difference as product:

(i) $\sin 5\theta + \sin 3\theta$ (ii) $\sin 8\theta - \sin 4\theta$ (iii) $\cos 6\theta + \cos 3\theta$

(iv) $\cos 7\theta - \cos\theta$ (v) $\cos 12^\circ + \cos 48^\circ$ (vi) $\sin(x+30^\circ) + \sin(x-30^\circ)$

Solution

(i) Since $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\sin 5\theta + \sin 3\theta = 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$

$$= 2\sin\left(\frac{8\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)$$

$$= 2\sin 4\theta\ \cos\theta$$

(ii) Since $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$$\sin 8\theta - \sin 4\theta = 2\cos\left(\frac{8\theta+4\theta}{2}\right)\sin\left(\frac{8\theta-4\theta}{2}\right)$$

$$= 2\cos 6\theta\ \sin 2\theta$$

(iii)

(iv) Since $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$$\cos 7\theta - \cos\theta = -2\sin\left(\frac{7\theta+\theta}{2}\right)\sin\left(\frac{7\theta-\theta}{2}\right)$$

$$= -2\sin\left(\frac{8\theta}{2}\right)\sin\left(\frac{6\theta}{2}\right) = -2\sin 4\theta\ \sin 3\theta$$

(v) Since $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\cos 12 + \cos 48 = 2\cos\left(\frac{12+48}{2}\right)\cos\left(\frac{12-48}{2}\right)$$

$$= 2\cos\left(\frac{60}{2}\right)\cos\left(\frac{-36}{2}\right)$$

$$= 2\cos 30\cos(-18)$$

$$= 2\cos 30^\circ\cos 18^\circ \qquad \because\ \cos(-\theta) = \cos\theta$$

(vi) Since $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\sin(x+30) + \sin(x-30) = 2\sin\left(\frac{x+30+x-30}{2}\right)\cos\left(\frac{x+30-x+30}{2}\right)$$

$$= 2\sin\left(\frac{2x}{2}\right)\cos\left(\frac{60}{2}\right) = 2\sin x\cos 30$$

Question # 3 Prove the following identities:

(i) $\dfrac{\sin 3x-\sin x}{\cos x-\cos 3x}=\cot 2x$ (ii) $\dfrac{\sin 8x+\sin 2x}{\cos 8x+\cos 2x}=\tan 5x$

(iii) $\dfrac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}=\cot\left(\dfrac{\alpha+\beta}{2}\right)\tan\left(\dfrac{\alpha-\beta}{2}\right)$

Solution

(i) L.H.S $=\dfrac{\sin 3x-\sin x}{\cos x-\cos 3x}$

$$=\frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}$$

$$=\frac{\cos\left(\frac{4x}{2}\right)\sin\left(\frac{2x}{2}\right)}{-\sin\left(\frac{4x}{2}\right)\sin\left(\frac{-2x}{2}\right)}$$

$$=\frac{\cos(2x)\sin(x)}{+\sin(2x)\sin(x)}$$

$=\cot 2x=$ R.H.S

(ii) L.H.S $=\dfrac{\sin 8x+\sin 2x}{\cos 8x+\cos 2x}$

$$=\frac{\cancel{2}\sin\left(\frac{8x+2x}{2}\right)\cancel{\cos\left(\frac{8x-2x}{2}\right)}}{\cancel{2}\cos\left(\frac{8x+2x}{2}\right)\cancel{\cos\left(\frac{8x-2x}{2}\right)}}$$

$$=\frac{\sin\left(\frac{10x}{2}\right)}{\cos\left(\frac{10x}{2}\right)}$$

$=\dfrac{\sin 5x}{\cos 5x}$

$=\tan 5x=$ R.H.S

(iii) L.H.S $=\dfrac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}$

$$=\frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$=\cot\left(\dfrac{\alpha+\beta}{2}\right)\tan\left(\dfrac{\alpha-\beta}{2}\right)=$ R.H.S

Question # 4 Prove that:

(i) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

(ii) $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{2}\cos 2\theta$

(iii) $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta$

Solution

(i) L.H.S $= \cos 20^\circ + \cos 100^\circ + \cos 140^\circ$

$$= (\cos 100^\circ + \cos 20^\circ) + \cos 140^\circ$$

$$= 2\cos\left(\frac{100+20}{2}\right)\cos\left(\frac{100-20}{2}\right) + \cos 140^\circ$$

$$= 2\cos 60^\circ \cos 40^\circ + \cos 140^\circ$$

$$= 2\left(\frac{1}{2}\right)\cos 40^\circ + \cos 140^\circ \qquad \cos 60^\circ = \frac{1}{2}$$

$$= \cos 140^\circ + \cos 40^\circ$$

$$= 2\cos\left(\frac{140+40}{2}\right)\cos\left(\frac{140-40}{2}\right)$$

$$= 2\cos 90^\circ \cos 50^\circ$$

$$= 2(0)\cos 50^\circ = 0 = \text{R.H.S} \qquad \cos 90^\circ = 0$$

(ii) L.H.S $= \sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)$

$$= \left(\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta\right)\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right)$$

$$= \left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)$$

$$= \left(\frac{1}{\sqrt{2}}\cos\theta\right)^2 - \left(\frac{1}{\sqrt{2}}\sin\theta\right)^2 = \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta$$

$$= \frac{1}{2}\left(\cos^2\theta - \sin^2\theta\right) = \frac{1}{2}\cos 2\theta = \text{R.H.S}$$

(iii) L.H.S $= \dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}$

$$= \frac{(\sin 7\theta+\sin\theta)+(\sin 5\theta+\sin 3\theta)}{(\cos 7\theta+\cos\theta)+(\cos 5\theta+\cos 3\theta)}$$

$$= \frac{2\sin\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}{2\cos\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\cos\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}$$

$$= \frac{2\sin 4\theta\cos 3\theta+2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta+2\cos 4\theta\cos\theta}$$

$$= \frac{2\sin 4\theta(\cos 3\theta+\cos\theta)}{2\cos 4\theta(\cos 3\theta+\cos\theta)} = \frac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta = \text{R.H.S}$$

Question # 5

Prove that:

(i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$ (ii) $\sin\dfrac{\pi}{9}\sin\dfrac{2\pi}{9}\sin\dfrac{\pi}{3}\sin\dfrac{4\pi}{9} = \dfrac{3}{16}$

Solution

(i) L.H.S $= \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$$= \cos 20^\circ \cos 40^\circ \left(\frac{1}{2}\right)\cos 80^\circ \quad = \frac{1}{2}\cos 80^\circ \cos 40^\circ \cos 20^\circ$$

$$= \frac{1}{4}\left(2\cos 80^\circ \cos 40^\circ\right)\cos 20^\circ \quad = \frac{1}{4}\left(\cos(80+40) + \cos(80-40)\right)\cos 20^\circ$$

$$= \frac{1}{4}\left(\cos 120^\circ + \cos 40^\circ\right)\cos 20^\circ \quad = \frac{1}{4}\left(-\frac{1}{2} + \cos 40^\circ\right)\cos 20^\circ$$

$$= \sin\frac{180^\circ}{9}\sin\frac{2(180^\circ)}{9}\sin\frac{(180^\circ)}{3}\sin\frac{4(180^\circ)}{9} \qquad \because\ \pi = 180^\circ$$

$$= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ \quad = \sin 20^\circ \sin 40^\circ \frac{\sqrt{3}}{2}\sin 80^\circ$$

$$= \frac{\sqrt{3}}{2}\sin 80^\circ \sin 40^\circ \sin 20^\circ \quad = -\frac{\sqrt{3}}{4}\left(-2\sin 80^\circ \sin 40^\circ\right)\sin 20^\circ$$

$$= -\frac{\sqrt{3}}{4}\left(\cos(80+40) - \cos(80-40)\right)\sin 20^\circ$$

$$= -\frac{\sqrt{3}}{4}\left(\cos 120^\circ - \cos 40^\circ\right)\sin 20^\circ \quad = -\frac{\sqrt{3}}{4}\left(-\frac{1}{2} - \cos 40^\circ\right)\sin 20^\circ$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{4}\cos 40^\circ \sin 20^\circ \quad = \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(2\cos 40^\circ \sin 20^\circ\right)$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(\sin(40+20) - \sin(40-20)\right)$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(\sin 60^\circ - \sin 20^\circ\right) \quad = \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{\sqrt{3}}{8}\left(\frac{\sqrt{3}}{2} - \sin 20^\circ\right)$$

$$= \frac{\sqrt{3}}{8}\sin 20^\circ + \frac{3}{16} - \frac{\sqrt{3}}{8}\sin 20^\circ \quad = \frac{3}{16} \quad = \text{R.H.S}$$

(ii)

$\downarrow$