Question # 1 Find the values of $\sin 2\alpha, \cos 2\alpha$ and $\tan 2\alpha$ when:

(i) $\sin\alpha = \frac{12}{13}$ (ii) $\cos\alpha = \frac{3}{5}$ where $0 < \alpha < \frac{\pi}{2}$

Solution

(i) $\sin\alpha = \frac{12}{13}$; $0 < \alpha < \frac{\pi}{2}$

$$\cos\alpha = \sqrt{1-\sin^2\alpha}$$

Since $\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$ As α is in the first quadrant so value of cos is +ive

$$= \sqrt{1-\left(\frac{12}{13}\right)^2}$$

$$= \sqrt{1-\frac{144}{169}}$$

$$= \sqrt{\frac{25}{169}} \Rightarrow \cos\alpha = \frac{5}{13}$$

and $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{12/13}{5/13} \Rightarrow \tan\alpha = \frac{12}{5}$

Now $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) \Rightarrow \boxed{\sin 2\alpha = \frac{120}{169}}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2$$

$$= \frac{25}{169} - \frac{144}{169} \Rightarrow \boxed{\cos 2\alpha = \frac{119}{169}}$$

$$\tan 2\alpha = \frac{2\alpha\tan}{1-\tan^2\alpha}$$

$$= \frac{2\left(12/5\right)}{1-\left(12/5\right)^2}$$

$$= \frac{24/5}{1-144/25}$$

$$= \frac{24/5}{-119/25}$$

$$= -\frac{24}{5}\cdot\frac{25}{119} \Rightarrow \boxed{\tan 2\alpha = \frac{120}{119}}$$

(iii) Let $\theta = 54^\circ$

$\Rightarrow \quad 5\theta = 270^\circ$

$\Rightarrow \quad 3\theta + 2\theta = 270^\circ$

$\Rightarrow \quad 2\theta = 270^\circ - 3\theta$

$\sin 2\theta = \sin(270 - 3\theta)$

$\Rightarrow \sin 2\theta = \sin(3(90) - 3\theta)$

$\Rightarrow \sin 2\theta = -\cos 3\theta$

$\Rightarrow 2\sin\theta\cos\theta = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow 2\sin\theta\cos\theta = -4\cos^3\theta + 3\cos\theta)$

$\Rightarrow 2\sin\theta = -4\cos^2\theta + 3$

$\Rightarrow 2\sin\theta = -4(1 - \sin^2\theta) + 3$

$\Rightarrow 2\sin\theta = -4 + 4\sin^2\theta + 3$

$\Rightarrow 2\sin\theta = 4\sin^2\theta - 1$

$\Rightarrow 4\sin^2\theta - 2\sin\theta - 1 = 0$

$\because \cos 3\theta = 4\cos^3\theta - 3\cos($

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$\because \theta = 54^\circ$

$$\sin\theta = \frac{1+\sqrt{5}}{4}$$

$$\Rightarrow \boxed{\sin 54^\circ = \frac{1+\sqrt{5}}{4}}$$

Now $\cos^2 54^\circ = 1 - \sin^2 54^\circ$

$$\Rightarrow \cos^2 54^\circ = 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2$$

$$\cos^2 54^\circ = 1 - \frac{\left(\sqrt{5}\right)^2 + 2\sqrt{5} + 1}{16}$$

$$= 1 - \frac{5 + 2\sqrt{5} + 1}{16}$$

$$= 1 - \frac{6 + 2\sqrt{5}}{16}$$

$$= \frac{16 - 6 - \sqrt{5}}{16}$$

$$= \frac{10 - 2\sqrt{5}}{16}$$

$$\Rightarrow \cos 54^\circ = \sqrt{\frac{10 - 2\sqrt{5}}{16}} \quad \Rightarrow$$

$$\boxed{\cos 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}}$$

(iii) Now $\sin(90-36) = \cos 36^\circ$ **Method** $\because \sin(90-\theta) = \cos\theta$

$\Rightarrow \sin 54^\circ = \cos 36^\circ$

And $\Rightarrow \boxed{\sin 54^\circ = \dfrac{1+\sqrt{5}}{4}}$

$\cos(90-36) = \sin 36^\circ$

$\Rightarrow \cos 54^\circ = \sin 36^\circ$

$\Rightarrow \boxed{\cos 54^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}}$

(iv) Now $\sin(90-18) = \cos 18^\circ$ $\because \sin(90-\theta) = \cos\theta$

$\Rightarrow \sin 72^\circ = \cos 18^\circ$

$\Rightarrow \boxed{\sin 72^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}}$

and $\cos(90-18) = \sin 18^\circ$

$\Rightarrow \cos 72^\circ = \sin 18^\circ$

$\Rightarrow \boxed{\cos 72^\circ = \dfrac{\sqrt{5}-1}{4}}$

(ii) $\cos\alpha = \frac{3}{5}$; $0 < \alpha < \frac{\pi}{2}$

Hint: *First find* $\sin\alpha$ *and* $\tan\alpha$ *then solve as above*

$\downarrow$

Question # 2 $\cot\alpha-\tan\alpha=2\cot2\alpha$

Solution

$$\text{L.H.S}=\cot\alpha-\tan\alpha$$
$$=\frac{\cos\alpha}{\sin\alpha}-\frac{\sin\alpha}{\cos\alpha}$$
$$=\frac{\cos^2\alpha-\sin^2\alpha}{\sin\alpha\cos\alpha}$$
$$=\frac{2(\cos^2\alpha-\sin^2\alpha)}{2\sin\alpha\cos\alpha}$$
$$=\frac{2\cos2\alpha}{\sin2\alpha}$$
$$=2\cot2\alpha=\text{R.H.S}$$

Question # 3 $\dfrac{\sin2\alpha}{1+\cos2\alpha}=\tan\alpha$

Solution

$$\text{L.H.S}=\frac{\sin2\alpha}{1+\cos2\alpha}$$
$$=\frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha}$$
$$=\frac{\sin\alpha}{\cos\alpha}$$
$$=\tan\alpha=\text{R.H.S}$$

$\because \cos^2\alpha=\dfrac{1+\cos2\alpha}{2}$

$\therefore 2\cos^2\alpha=1+\cos2\alpha$

Question # 4 $\dfrac{1-\cos\alpha}{\sin\alpha}=\tan\dfrac{\alpha}{2}$

Solution

$$\text{L.H.S}=\frac{1-\cos\alpha}{\sin\alpha}$$
$$=\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$
$$=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$
$$=\tan\frac{\alpha}{2}=\text{R.H.S}$$

$\because \sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2}$

$\because 2\sin^2\dfrac{\alpha}{2}=1-\cos\alpha$

$\sin\alpha=2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}$

Question # 5 $\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}=\sec2\alpha-\tan2\alpha$

Solution

$$\text{L.H.S}=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}$$
$$=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\times\frac{\cos\alpha-\sin\alpha}{\cos\alpha-\sin\alpha}$$
$$=\frac{(\cos\alpha-\sin\alpha)^2}{\cos^2\alpha-\sin^2\alpha}$$
$$=\frac{\cos^2\alpha-\sin^2\alpha-2\sin\alpha\cos\alpha}{\cos2\alpha}$$
$$=\frac{1-\sin2\alpha}{\cos2\alpha}$$
$$=\frac{1}{\cos2\alpha}-\frac{\sin2\alpha}{\cos2\alpha}$$
$$=\sec2\alpha-\tan2\alpha=\text{R.H.S}$$

Question # 6 $\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}} = \dfrac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}$

Solution

$$\text{L.H.S} = \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$$
$$= \sqrt{\frac{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$
$$= \sqrt{\frac{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\right)^2}}$$

$\because \sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}=1$

$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$= \frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}} = \text{R.H.S}$$

Question # 7 $\dfrac{\operatorname{cosec}\theta+2\operatorname{cosec}2\theta}{\sec\theta} = \cot\dfrac{\theta}{2}$

Solution

$$\text{L.H.S} = \frac{\operatorname{cosec}\theta+2\operatorname{cosec}2\theta}{\sec\theta}$$
$$= \frac{\frac{1}{\sin\theta}+\frac{2}{\sin 2\theta}}{\frac{1}{\cos\theta}}$$
$$= \cos\theta\left(\frac{1}{\sin\theta}+\frac{2}{2\sin\theta\cos\theta}\right)$$
$$= \cos\theta\left(\frac{1}{\sin\theta}+\frac{1}{\sin\theta\cos\theta}\right)$$
$$= \cos\theta\left(\frac{\cos\theta+1}{\sin\theta\cos\theta}\right)$$
$$= \frac{\cos\theta+1}{\sin\theta}$$
$$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$
$$= \cot\frac{\theta}{2} = \text{R.H.S}$$

Question # 8 $1+\tan\alpha\tan 2\alpha = \sec 2\alpha$

Solution

$$\text{L.H.S} = 1+\tan\alpha\tan 2\alpha$$
$$= 1+\left(\frac{\sin\alpha}{\cos\alpha}\right)\left(\frac{\sin 2\alpha}{\cos 2\alpha}\right)$$
$$= \frac{\cos\alpha\cos 2\alpha+\sin\alpha\sin 2\alpha}{\cos\alpha\cos 2\alpha}$$
$$= \frac{\cos(2\alpha-\alpha)}{\cos\alpha\cos 2\alpha}$$
$$= \frac{\cos\alpha}{\cos\alpha\cos 2\alpha}$$
$$= \frac{1}{\cos 2\alpha}$$
$$= \sec 2\alpha = \text{R.H.S}$$

Question # 9 $\dfrac{2\sin\theta\sin2\theta}{\cos\theta+\cos3\theta}=\tan2\theta\tan\theta$

Solution

$$\text{L.H.S}=\frac{2\sin\theta\sin2\theta}{\cos\theta+\cos3\theta}$$
$$=\frac{2\sin\theta\sin2\theta}{\cos\theta+4\cos^3\theta-3\cos\theta}$$
$$=\frac{2\sin\theta\sin2\theta}{4\cos^3\theta-2\cos\theta}$$
$$=\frac{2\sin\theta\sin2\theta}{2\cos\theta(2\cos^2\theta-1)}$$
$$=\frac{\sin\theta\sin2\theta}{\cos\theta\cos2\theta}$$
$$=\tan\theta\tan2\theta$$
$$=\tan2\theta\tan\theta\quad=\text{R.H.S}$$

$\because\ \cos3\theta=4\cos^3\theta-3\cos\theta$

$\because\ \cos^2\theta=\dfrac{1+\cos2\theta}{2}$

$\therefore\ 2\cos^2\theta-1=\cos2\theta$

Question # 10 $\dfrac{\sin3\theta}{\sin\theta}-\dfrac{\cos3\theta}{\cos\theta}=2$

Solution

$$\text{L.H.S}=\frac{\sin3\theta}{\sin\theta}-\frac{\cos3\theta}{\cos\theta}$$
$$=\frac{\sin3\theta\cos\theta-\cos3\theta\sin\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$
$$=\frac{\sin2\theta}{\sin\theta\cos\theta}$$
$$=\frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$
$$=2=\text{R.H.S}$$

Question # 11 $\dfrac{\cos3\theta}{\cos\theta}+\dfrac{\sin3\theta}{\sin\theta}=4\cos2\theta$

Solution

$$\text{L.H.S}=\frac{\cos3\theta}{\cos\theta}+\frac{\sin3\theta}{\sin\theta}$$
$$=\frac{\cos3\theta\sin\theta+\sin3\theta\cos\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin(\theta+3\theta)}{\sin\theta\cos\theta}$$
$$=\frac{\sin4\theta}{\sin\theta\cos\theta}$$
$$=\frac{2\sin2\theta\cos2\theta}{\sin\theta\cos\theta}$$
$$=\frac{2(2\sin\theta\cos\theta)\cos2\theta}{\sin\theta\cos\theta}$$
$$=4\cos2\theta\quad=\text{R.H.S}$$

Question # 12 $\dfrac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}=\sec\theta$

Solution

$$\text{L.H.S} = \frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}$$

$$= \frac{\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}+\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}}{\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}-\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{\dfrac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}{\dfrac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$$

$$= \frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta = \text{R.H.S}$$

Question # 13 $\dfrac{\sin 3\theta}{\cos\theta}+\dfrac{\cos 3\theta}{\sin\theta}=2\cot 2\theta$

Solution

$$\text{L.H.S} = \frac{\sin 3\theta}{\cos\theta}+\frac{\cos 3\theta}{\sin\theta}$$

$$= \frac{\sin 3\theta\sin\theta+\cos 3\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos(3\theta-\theta)}{\sin\theta\cos\theta}$$

$$= \frac{\cos 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos 2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\cos 2\theta}{\sin 2\theta}$$

$$= 2\cot 2\theta = \text{R.H.S}$$

Question # 14 Reduce $\sin^4\theta$ to an expression involving only functions of multiples of θ raised to the first power.

Solution

$$\sin^4\theta = \left(\sin^2\theta\right)^2$$
$$= \left(\frac{1-\cos 2\theta}{2}\right)^2$$
$$= \frac{1-2\cos 2\theta + \cos^2 2\theta}{4}$$
$$= \frac{1}{4}\left(1-2\cos 2\theta + \cos^2 2\theta\right)$$
$$= \frac{1}{4}\left(1-2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$
$$= \frac{1}{4}\left(\frac{2-4\cos 2\theta + 1 + \cos 4\theta}{2}\right)$$
$$= \frac{1}{8}\left(3-4\cos 2\theta + \cos 4\theta\right)$$

Question # 15 Find the values of $\sin\theta$ and $\cos\theta$, without using table or calculator, when θ

(i) 18° (ii) 36° (iii) 54° (iv) 72°

Solution

(i) Let $\theta = 18^\circ$

$\Rightarrow \quad 5\theta = 90^\circ$

$\Rightarrow \quad 3\theta + 2\theta = 90^\circ$

$\Rightarrow \quad 2\theta = 90^\circ - 3\theta$

$\sin 2\theta = \sin(90 - 3\theta)$

$\Rightarrow \sin 2\theta = \cos 3\theta$

$\Rightarrow 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$\because \cos 3\theta = 4\cos^3\theta - 3$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow 2\sin\theta = 4\cos^2\theta - 3$

$\Rightarrow 2\sin\theta = 4(1 - \sin^2\theta) - 3$

$\Rightarrow 2\sin\theta = 4 - 4\sin^2\theta - 3$

$\Rightarrow 2\sin\theta = 1 - 4\sin^2\theta$

$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

This is quadratic in $\sin\theta$

$a = 4$, $b = 1$ and $c = -1$

$$\sin\theta = \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2(4)}$$

$$= \frac{-2 \pm \sqrt{4+16}}{8}$$

$$= \frac{-2 \pm \sqrt{20}}{8}$$

$$= \frac{-2 \pm 2\sqrt{5}}{8}$$

$$= \frac{-1 \pm \sqrt{5}}{8}$$

$$\sin\theta = \frac{-1 + \sqrt{5}}{4}$$

Since $\theta = 18^\circ$ lies in the first quadrant so value of sin can not be negative therefore

$\because \theta = 18^\circ$

$$\Rightarrow \boxed{\sin 18^\circ = \frac{\sqrt{5} - 1}{4}}$$

Now $\cos^2 18^\circ = 1 - \sin^2 18^\circ$

$$\Rightarrow \cos^2 18^\circ = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2$$

$$\Rightarrow \cos^2 18^\circ = 1 - \frac{(\sqrt{5})^2 - 2\sqrt{5} + 1}{16}$$

$$= 1 - \frac{5 - 2\sqrt{5} + 1}{16}$$

$$= 1 - \frac{6 - 2\sqrt{5}}{16}$$

$$= \frac{16 - 6 + \sqrt{5}}{16}$$

$$= \frac{10 + 2\sqrt{5}}{16}$$

$$\Rightarrow \cos 18^\circ = \sqrt{\frac{10 + 2\sqrt{5}}{16}} \quad \Rightarrow \quad \boxed{\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}}$$

(ii) Since $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$2\cos^2\theta = 1+\cos 2\theta$$

$$\Rightarrow \quad \cos 2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \quad \cos 2(18) = 2\cos^2(18) - 1$$

$$\Rightarrow \quad \cos 36 = 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1$$

$$= 2\left(\frac{10+2\sqrt{5}}{16}\right) - 1$$

$$= \frac{10+2\sqrt{5}}{8} - 1$$

$$= \frac{10+2\sqrt{5}-8}{8}$$

$$= \frac{2+2\sqrt{5}}{8}$$

$$\Rightarrow \quad \boxed{\cos 36^\circ = \frac{1+\sqrt{5}}{4}}$$

$$\sin^2 36 = 1 - \cos^2 36$$

$$= 1 - \left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$= 1 - \frac{1+2\sqrt{5}+\left(\sqrt{5}\right)^2}{16}$$

$$= 1 - \frac{1+2\sqrt{5}+5}{16}$$

$$= 1 - \frac{6+2\sqrt{5}}{16}$$

Now

$$= \frac{16-6-2\sqrt{5}}{16}$$

$$= \frac{10-2\sqrt{5}}{16}$$

$$\Rightarrow \quad \sin 36^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}}$$

$$\Rightarrow \quad \boxed{\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$