Systems of linear equations (Chapter No. 4)

Mathematical Method

Consider m linear eqs. in n unknowns

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= b_m
\end{aligned}\right]$$

The above system of linear eqs. can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is the matrix of Cofficients of Variables.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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Note Let the system $AX = B$ is given.

① If $B \neq 0$
Then this system is called non homogeneous system of linear eqs.

② If $B = 0$
then $AX = 0$
& then this system is called homogeneous system of linear eqs.

③ If the system $AX = B$ has soln. Then this system is called consistent.

④ If the system $AX = B$ has no soln. then this system is called inconsistent.

Type ① when no. of eqs. is equal to the no. of variables & system $AX = B$ is non homogeneous then unique soln. of the system exists if matrix A is non singular after applying row operations.

Type ② When no. of eqs. is not equal (may be equal) to the no. of variables & system is non homogenous Then the system has a soln. if

$$\text{rank } A = \text{rank } A_b$$

Type ③ A system of m homogeneous linear eqs. $AX = 0$ in n unknowns has a non trivial soln. if

$$\text{rank } A < n$$

where n is no. of columns of A

Guassian elimination method

In this method we reduce the augmented matrix into echelon form. In this way, the value of last variable is calculated & then by backward substitution, the values of remaining unknowns can be calculated.

Guass Jordan method:

In this method, we reduce the augmented matrix into reduced echelon form by applying row operations. In this way, the values of all the unknowns is calculated directly without any backward substitution.

※ Exercise No. 4 ※

Solve the following systems of linear equations, the field of scalars being R:

Mathematical Method

Q1
$$2x_1 + x_3 = 1$$
$$2x_1 + 4x_2 - x_3 = -2$$
$$x_1 - 8x_2 - 3x_3 = 2$$

Sol: Given system is

$$2x_1 + x_3 = 1$$
$$2x_1 + 4x_2 - x_3 = -2$$
$$x_1 - 8x_2 - 3x_3 = 2$$

Take augmented matrix

$$A_b = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 4 & -1 & -2 \\ 1 & -8 & -3 & 2 \end{bmatrix}$$

We reduce A to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & -8 & -3 & 2 \\ 2 & 4 & -1 & -2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -8 & -3 & 2 \\ 0 & 20 & 5 & -6 \\ 0 & 16 & 7 & -3 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -8 & -3 & 2 \\ 0 & 1 & \frac{1}{4} & -\frac{3}{10} \\ 0 & 16 & 7 & -3 \end{bmatrix} \qquad \frac{1}{20}R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -1 & -\frac{2}{5} \\ 0 & 1 & \frac{1}{4} & -\frac{3}{10} \\ 0 & 0 & 3 & \frac{9}{5} \end{bmatrix} \qquad \begin{matrix} R_3 - 16R_2 \\ R_1 + 8R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -1 & -2/5 \\ 0 & 1 & 1/4 & -3/10 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} \qquad \frac{1}{3}R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & -9/20 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} \qquad \begin{matrix} R_1 + R_3 \\ R_2 - \frac{1}{4}R_3 \end{matrix}$$

Since matrix A is non singular
So unique solution exists
& soln. is

$$\left.\begin{aligned} x_1 &= \frac{1}{5} \\ x_2 &= -\frac{9}{20} \\ x_3 &= \frac{3}{5} \end{aligned}\right\}$$

Q2 $x_1 + x_2 + x_3 = a$

$x_1 + (1+a)x_2 + x_3 = 2a$ where $a \neq 0$

$x_1 + x_2 + (1+a)x_3 = 3a$

Sol:- Given system is

$$x_1 + x_2 + x_3 = a$$
$$x_1 + (1+a)x_2 + x_3 = 2a$$
$$x_1 + x_2 + (1+a)x_3 = 3a$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 1+a & 1 & 2a \\ 1 & 1 & 1+a & 3a \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & a & 0 & a \\ 0 & 0 & a & 2a \end{bmatrix} \qquad \begin{matrix} R_2-2R_1 \\ R_3-R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & 2a \end{bmatrix} \qquad \frac{1}{a}R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1 & a-1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & 2a \end{bmatrix} \qquad R_1-R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1 & a-1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \frac{1}{a}R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & a-3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1-R_3$$

As matrix A is non singular

So unique solution exists

& soln. is

$$x_1 = a-3$$
$$x_2 = 1$$
$$x_3 = 2$$

Q3
$$\begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 &= 1 \\ 2x_1 + x_2 + 3x_3 \quad + 4x_5 &= 3 \\ 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1 \\ x_2 + x_4 + x_5 &= 0 \end{aligned}$$

Sol. Given system is

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 &= 1 \\ 2x_1 + x_2 + 3x_3 \quad + 4x_5 &= 3 \\ 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1 \\ x_2 + x_4 + x_5 &= 0 \end{aligned}$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 2 & 1 & 3 & 0 & 4 & 3 \\ 3 & -2 & 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 3 & 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 3 & 1 & 2 & 2 & 1 \end{bmatrix} \qquad R_{24}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 & -3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \qquad \begin{matrix} R_1 + R_2 \\ R_3 - R_2 \\ R_4 - 3R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \qquad (-1)R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 2 \\ 0 & 0 & 0 & 2 & -4 & -1 \end{bmatrix} \qquad R_4 - R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1/2 \end{bmatrix} \qquad \tfrac{1}{2}R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc:c:c} 1 & 0 & 0 & 0 & 5 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 3 & 1/2 \\ 0 & 0 & 1 & 0 & -3 & 1/2 \\ 0 & 0 & 0 & 1 & -2 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 - R_4 \\ R_2 - R_4 \\ R_3 + R_4 \end{array}$$

Here rank A = rank A_b

So soln. exists & soln. is

$$\left.\begin{array}{rcl} x_1 + 5x_5 & = & \frac{1}{2} \\ x_2 + 3x_5 & = & \frac{1}{2} \\ x_3 - 3x_5 & = & \frac{1}{2} \\ x_4 - 2x_5 & = & -1/2 \end{array}\right\}$$

So req. soln. is

$$\left.\begin{array}{rcl} x_1 & = & \frac{1}{2} - 5x_5 \\ x_2 & = & \frac{1}{2} - 3x_5 \\ x_3 & = & \frac{1}{2} + 3x_5 \\ x_4 & = & -\frac{1}{2} + 2x_5 \end{array}\right\}$$ where x_5 is arbitrary.

Q4

$$\begin{array}{rcl} x_1 + x_2 - x_3 & = & 1 \\ x_2 + x_3 - x_4 & = & 1 \\ x_3 + x_4 - x_5 & = & 1 \\ x_5 + x_4 - x_3 & = & 1 \\ x_4 + x_3 - x_2 & = & 1 \end{array}$$

Sol. Given system is

$$\begin{array}{rcl} x_1 + x_2 - x_3 & = & 1 \\ x_2 + x_3 - x_4 & = & 1 \\ x_3 + x_4 - x_5 & = & 1 \\ x_5 + x_4 - x_3 & = & 1 \\ x_4 + x_3 - x_2 & = & 1 \end{array}$$

Take augmented matrix.

$$A_b = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{matrix} R_1 - R_2 \\ R_5 + R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 & 2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{bmatrix} \qquad \begin{matrix} R_1 + 2R_3 \\ R_2 - R_3 \\ R_4 + R_3 \\ R_5 - 2R_3 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 & 2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{bmatrix} \qquad \tfrac{1}{2} R_4$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix} \qquad \begin{matrix} R_1 - 3R_4 \\ R_2 + 2R_4 \\ R_3 - R_4 \\ R_5 + 2R_4 \end{matrix}$$

$$\sim_R \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \frac{1}{2}R_5$$

$$\sim_R \left[\begin{array}{ccccc:c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Since matrix A is non singular, so unique soln. exists & soln. is

$$\left.\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 1 \\ x_4 &= 1 \\ x_5 &= 1 \end{aligned}\right\}$$

Q5

$$\begin{aligned} x_1 - 2x_2 - 7x_3 + 7x_4 &= 5 \\ -x_1 + 2x_2 + 8x_3 - 5x_4 &= -7 \\ 3x_1 - 4x_2 - 17x_3 + 13x_4 &= 14 \\ 2x_1 - 2x_2 - 11x_3 + 8x_4 &= 7 \end{aligned}$$

Sol:- Given system is

$$\begin{aligned} x_1 - 2x_2 - 7x_3 + 7x_4 &= 5 \\ -x_1 + 2x_2 + 8x_3 - 5x_4 &= -7 \\ 3x_1 - 4x_2 - 17x_3 + 13x_4 &= 14 \\ 2x_1 - 2x_2 - 11x_3 + 8x_4 &= 7 \end{aligned}$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ -1 & 2 & 8 & -5 & -7 \\ 3 & -4 & -17 & 13 & 14 \\ 2 & -2 & -11 & 8 & 7 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 2 & 4 & -8 & -1 \\ 0 & 2 & 3 & -6 & -3 \end{bmatrix} \qquad \begin{matrix} R_2 + 2R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ 0 & 2 & 4 & -8 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 2 & 3 & -6 & -3 \end{bmatrix} \qquad R_{23}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ 0 & 1 & 2 & -4 & -1/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 2 & 3 & -6 & -3 \end{bmatrix} \qquad \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 0 & -3 & -1 & 4 \\ 0 & 1 & 2 & -4 & -1/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & -1 & 2 & -2 \end{bmatrix} \qquad \begin{matrix} R_1 + 2R_2 \\ R_4 - 2R_2 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 & -2 \\ 0 & 1 & 0 & -8 & 7/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \qquad \begin{matrix} R_1 + 3R_3 \\ R_2 - 2R_3 \\ R_4 + R_3 \end{matrix}$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 5 & -2 \\ 0 & 1 & 0 & -8 & 7/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \frac{1}{4}R_4$$

$$\underset{R}{\sim}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R_1 - 5R_4 \\ R_2 + 8R_4 \\ R_3 - 2R_4 \end{array}$$

Since matrix A is non singular
So unique soln. exists & soln. is

$$\left.\begin{array}{l} x_1 = 3 \\ x_2 = -9/2 \\ x_3 = 0 \\ x_4 = -1 \end{array}\right\}$$

Q6
$$x_1 + 2x_2 + x_3 = -1$$
$$6x_1 + x_2 + x_3 = -4$$
$$2x_1 - 3x_2 - x_3 = 0$$
$$x_1 - x_2 = 1$$

Sol. Given system is

$$x_1 + 2x_2 + x_3 = -1$$
$$6x_1 + x_2 + x_3 = -4$$
$$2x_1 - 3x_2 - x_3 = 0$$
$$x_1 - x_2 = 1$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 6 & 1 & 1 & -4 \\ 2 & -3 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & -11 & -5 & 2 \\ 0 & -7 & -3 & 2 \\ 0 & -3 & -1 & 2 \end{bmatrix} \qquad \begin{matrix} R_2 - 6R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & -6 \\ 0 & -7 & -3 & 2 \\ 0 & -3 & -1 & 2 \end{bmatrix} \qquad R_2 - 4R_4$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 3 & 11 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & -10 & -40 \\ 0 & 0 & -4 & -16 \end{bmatrix} \qquad \begin{matrix} R_1 - 2R_2 \\ R_3 + 7R_2 \\ R_4 + 3R_2 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 3 & 11 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -4 & -16 \end{bmatrix} \qquad -\frac{1}{10} R_3$$

$$\overset{R}{\sim} \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{matrix} R_1 - 3R_3 \\ R_2 + R_3 \end{matrix}$$

As rank A = rank A_b. So soln. exists & soln. is

$$\left.\begin{aligned} x_1 &= -1 \\ x_2 &= -2 \\ x_3 &= 4 \end{aligned}\right\}$$

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Q7
$$2x_1 + x_2 + 5x_3 = 4$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$5x_1 - 8x_2 - 4x_3 = 5$$

Sol. Given system is
$$2x_1 + x_2 + 5x_3 = 4$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$5x_1 - 8x_2 - 4x_3 = 5$$

Take augmented matrix

$$A_b = \begin{bmatrix} 2 & 1 & 5 & 4 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 5 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations

$$\underset{\sim}{R} \begin{bmatrix} 2 & 1 & 5 & 4 \\ 1 & -3 & -3 & -2 \\ 5 & -8 & -4 & 5 \end{bmatrix} \quad R_2 - R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -3 & -3 & -2 \\ 2 & 1 & 5 & 4 \\ 5 & -8 & -4 & 5 \end{bmatrix} \quad R_{12}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -3 & -3 & -2 \\ 0 & 7 & 11 & 8 \\ 0 & 7 & 11 & 15 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -3 & -3 & -2 \\ 0 & 1 & \frac{11}{7} & \frac{8}{7} \\ 0 & 7 & 11 & 15 \end{bmatrix} \quad \frac{1}{7}R_2$$

$$\sim\begin{bmatrix} 1 & 0 & \frac{13}{7} & \frac{12}{7} \\ 0 & 1 & \frac{11}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 7 \end{bmatrix} \qquad \begin{matrix} R_1 + 3R_2 \\ R_3 - 7R_2 \end{matrix}$$

Since rank $A \neq$ rank $\tilde{A}b$

So soln. does not exists

For what value of λ have the following homogeneous equations non trivial solutions? Find these solns (Prob 8–10)

Q8 $(1-\lambda)x_1 + x_2 = 0$

$x_1 + (1-\lambda)x_2 = 0$

Sol. Given system is

$$(1-\lambda)x_1 + x_2 = 0$$
$$x_1 + (1-\lambda)x_2 = 0$$

Take matrix A of Co-efficients of Variables

$$A = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1-\lambda \\ 1-\lambda & 1 \end{bmatrix} \qquad R_{12}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1-\lambda \\ 0 & 2\lambda-\lambda^2 \end{bmatrix} \longrightarrow (A) \qquad R_2 - (1-\lambda)R_1$$

For non trivial soln., rank $A < 2$

$$\Rightarrow 2\lambda - \lambda^2 = 0$$
$$\lambda(2-\lambda) = 0$$
$$\Rightarrow \boxed{\lambda = 0, 2}$$

Put $\lambda = 0$ in (A)

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 + x_2 = 0$$

or $x_2 = -x_1$

So for $\lambda = 0$, non trivial soln. is $x_2 = -x_1$; x_1 is arbitrary

Now Put $\lambda = 2$ in (A)

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 - x_2 = 0$$

or $x_2 = x_1$

So for $\lambda = 2$, non trivial soln. is $x_2 = x_1$; x_1 is arbitrary

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Q9 $(3-\lambda)x_1 - x_2 + x_3 = 0$

$x_1 - (1-\lambda)x_2 + x_3 = 0$

$x_1 - x_2 + (1-\lambda)x_3 = 0$

Soln. Given system is

$$(3-\lambda)x_1 - x_2 + x_3 = 0$$

$$x_1 - (1-\lambda)x_2 + x_3 = 0$$

$$x_1 - x_2 + (1-\lambda)x_3 = 0$$

Take matrix A of Cofficients of Variables

$$A = \begin{bmatrix} 3-\lambda & -1 & 1 \\ 1 & -(1-\lambda) & 1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations only.

$$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 1 & -(1-\lambda) & 1 \\ 3-\lambda & -1 & 1 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda & \lambda \\ 0 & 2-\lambda & -\lambda^2+4\lambda-2 \end{bmatrix} \longrightarrow (\alpha) \qquad \begin{matrix} R_2 - R_1 \\ R_3-(3-\lambda)R_1 \end{matrix}$$

Suppose $\lambda \neq 0$

So $$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & 1 & 1 \\ 0 & 2-\lambda & -\lambda^2+4\lambda-2 \end{bmatrix} \qquad \frac{1}{\lambda}R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 2-\lambda \\ 0 & 1 & 1 \\ 0 & 0 & -\lambda^2+5\lambda-4 \end{bmatrix} \longrightarrow (A) \qquad \begin{matrix} R_1+R_2 \\ R_3+(\lambda-2)R_2 \end{matrix}$$

For non trivial soln.

rank A $<$ 3 (no. of columns)

$$\Rightarrow \quad -\lambda^2+5\lambda-4 = 0$$

$$\lambda^2-5\lambda+4 = 0$$

$$(\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \quad \boxed{\lambda = 1, 4}$$

Put $\lambda = 1$ in matrix (A)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 + x_3 = 0$$

$$x_2 + x_3 = 0$$

or $x_1 = -x_3$

$x_2 = -x_3$

So for $\lambda = 1$, non trivial soln. is

$$\left.\begin{array}{l} x_1 = -x_3 \\ x_2 = -x_3 \end{array}\right\}$$ where x_3 is arbitrary

Now put $\lambda = 4$ in matrix (A)

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 - 2x_3 = 0$$

$$x_2 + x_3 = 0$$

or $x_1 = 2x_3$

$x_2 = -x_3$

Hence for $\lambda = 4$, non trivial soln. is

$$\left.\begin{array}{l} x_1 = 2x_3 \\ x_2 = -x_3 \end{array}\right\}$$ where x_3 is arbitrary

In case $\lambda = 0$, put in matrix (A)

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_{23}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \tfrac{1}{2} R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 + R_2$$

So non trivial soln. is

$$x_2 - x_3 = 0$$

or $x_2 = x_3$

Hence for $\lambda = 0$, non trivial soln. is

$$\left.\begin{array}{l} x_1 = 0 \\ x_2 = x_3 \end{array}\right\}$$ where x_3 is arbitrary

---

Q10
$$(1-\lambda)x_1 + x_2 + x_3 = 0$$
$$x_1 - \lambda x_2 + x_3 = 0$$
$$x_1 - x_2 + (1-\lambda)x_3 = 0$$

Sol. Given system is

$$(1-\lambda)x_1 + x_2 + x_3 = 0$$
$$x_1 - \lambda x_2 + x_3 = 0$$
$$x_1 - x_2 + (1-\lambda)x_3 = 0$$

Take matrix A of Cofficients of Variables

$$A = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1-\lambda \\ 1 & -\lambda & 1 \\ 1-\lambda & 1 & 1 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & 1-\lambda & \lambda \\ 0 & -\lambda & 2\lambda-\lambda^2 \end{bmatrix} \qquad \begin{array}{l} R_2 - R_1 \\ R_3-(1-\lambda)R_1 \end{array}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & 1 & -\lambda+\lambda^2 \\ 0 & -\lambda & 2\lambda-\lambda^2 \end{bmatrix} \qquad R_2 - R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1-2\lambda+\lambda^2 \\ 0 & 1 & -\lambda+\lambda^2 \\ 0 & 0 & \lambda^3-2\lambda^2+2\lambda \end{bmatrix} \longrightarrow A \qquad \begin{matrix} R_1+R_2 \\ R_3+\lambda R_2 \end{matrix}$$

For non trivial soln.

rank A < 3 (no. of Columns)

$$\Rightarrow \quad \lambda^3 - 2\lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 2) = 0$$

$$\lambda = 0 \quad , \quad \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

take only real value of λ

So $\boxed{\lambda = 0}$

Put in last matrix A

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 + x_3 = 0$$

$$x_2 = 0$$

So for $\lambda = 0$, non trivial soln. is

$$\left.\begin{matrix} x_1 = -x_3 \\ x_2 = 0 \end{matrix}\right\} \text{ where } x_3 \text{ is arbitrary}$$

In each of the following cases, use Guass Jordan method to reduce the given system to the reduced echelon form, indicating the operations you perform & determine the soln. if any.
(Problems 11 to 18) :

Q11
$$6x_1 - 6x_2 + 6x_3 = 6$$
$$2x_1 - 4x_2 - 6x_3 = 12$$
$$10x_1 - 5x_2 + 5x_3 = 30$$

Sol. Given system is
$$6x_1 - 6x_2 + 6x_3 = 6$$
$$2x_1 - 4x_2 - 6x_3 = 12$$
$$10x_1 - 5x_2 + 5x_3 = 30$$

Take augmented matrix

$$A_b = \begin{bmatrix} 6 & -6 & 6 & 6 \\ 2 & -4 & -6 & 12 \\ 10 & -5 & 5 & 30 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations.

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -2 & -3 & 6 \\ 2 & -1 & 1 & 6 \end{bmatrix} \qquad \frac{1}{6}R_1,\ \frac{1}{2}R_2,\ \frac{1}{5}R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & -4 & 5 \\ 0 & 1 & -1 & 4 \end{bmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & -1 & -4 & 5 \end{bmatrix} \qquad R_{23}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -5 & 9 \end{bmatrix} \qquad \begin{matrix} R_1+R_2 \\ R_3+R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -9/5 \end{bmatrix} \qquad -\frac{1}{5}R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 11/5 \\ 0 & 0 & 1 & -9/5 \end{array}\right] \qquad R_2+R_3$$

Since matrix A is non singular

So unique soln. exists & soln. is

$$\left.\begin{aligned} x_1 &= 5 \\ x_2 &= \frac{11}{5} \\ x_3 &= -\frac{9}{5} \end{aligned}\right\}$$

Q12
$$5x_1 - 2x_2 + x_3 = 0$$
$$3x_1 + 2x_2 + 7x_3 = 0$$
$$x_1 + x_2 + 3x_3 = 0$$

Sol. Given system is

$$5x_1 - 2x_2 + x_3 = 0$$
$$3x_1 + 2x_2 + 7x_3 = 0$$
$$x_1 + x_2 + 3x_3 = 0$$

Take matrix A of Cofficients of Variables

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 2 & 7 \\ 1 & 1 & 3 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 3\\ 3 & 2 & 7\\ 5 & -2 & 1\end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 3\\ 0 & -1 & -2\\ 0 & -7 & -14\end{bmatrix} \qquad \begin{matrix}R_2-3R_1\\ R_3-5R_1\end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 3\\ 0 & 1 & 2\\ 0 & -7 & -14\end{bmatrix} \qquad (-1)R_2$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix} \qquad \begin{matrix}R_1-R_2\\ R_3+7R_2\end{matrix}$$

Given system is equivalent to

$$x_1 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$

Hence infinite solutions of given system are

$$\left.\begin{matrix}x_1 = -x_3\\ x_2 = -2x_3\end{matrix}\right\} \text{ where } x_3 \text{ is arbitrary}$$

Q13
$$5x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + 2x_2 + 7x_3 = 5$$
$$x_1 + x_2 + 3x_3 = 2$$

Sol. Given system is

$$5x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + 2x_2 + 7x_3 = 5$$
$$x_1 + x_2 + 3x_3 = 2$$

Take augmented matrix

$$A_b = \begin{bmatrix}5 & -2 & 1 & 3\\ 3 & 2 & 7 & 5\\ 1 & 1 & 3 & 2\end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 2 & 7 & 5 \\ 5 & -2 & 1 & 3 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & -7 & -14 & -7 \end{bmatrix} \qquad \begin{matrix} R_2-3R_1 \\ R_3-5R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -7 & -14 & -7 \end{bmatrix} \qquad (-1)R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} R_1-R_2 \\ R_3+7R_2 \end{matrix}$$

Since rank A = rank A_b

So soln. exists & soln. is

$$x_1 + x_3 = 1$$
$$x_2 + 2x_3 = 1$$

or

$$\left.\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 1 - 2x_3 \end{aligned}\right\} \text{ where } x_3 \text{ is arbitrary.}$$

Q14
$$5x_1 - 2x_2 + x_3 = 2$$
$$3x_1 + 2x_2 + 7x_3 = 3$$
$$x_1 + x_2 + 3x_3 = 2$$

Sol:- Given system is
$$5x_1 - 2x_2 + x_3 = 2$$
$$3x_1 + 2x_2 + 7x_3 = 3$$
$$x_1 + x_2 + 3x_3 = 2$$

We reduce it to reduced echelon form by
applying row operations.

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 2 & 7 & 3 \\ 5 & -2 & 1 & 2 \end{bmatrix} \qquad R_{13}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & -7 & -14 & -8 \end{bmatrix} \qquad \begin{matrix} R_2-3R_1 \\ R_3-5R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & -7 & -14 & -8 \end{bmatrix} \qquad (-1)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 13 \end{bmatrix} \qquad \begin{matrix} R_1-R_2 \\ R_3+7R_2 \end{matrix}$$

Since rank A $\neq$ rank Ab
So soln. does not exist

Q15
$$2x_1 - x_2 + 3x_3 = 3$$
$$3x_1 + x_2 - 5x_3 = 0$$
$$4x_1 - x_2 + x_3 = 3$$

Soln. Given system is

$$2x_1 - x_2 + 3x_3 = 3$$
$$3x_1 + x_2 - 5x_3 = 0$$
$$4x_1 - x_2 + x_3 = 3$$

Take augmented matrix

$$Ab = \begin{bmatrix} 2 & -1 & 3 & 3 \\ 3 & 1 & -5 & 0 \\ 4 & -1 & 1 & 3 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim}\begin{bmatrix} -1 & -2 & 8 & 3 \\ 3 & 1 & -5 & 0 \\ 4 & -1 & 1 & 3 \end{bmatrix} \qquad R_1 - R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 3 & 1 & -5 & 0 \\ 4 & -1 & 1 & 3 \end{bmatrix} \qquad (-1)R_1$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 0 & -5 & 19 & 9 \\ 0 & -9 & 33 & 15 \end{bmatrix} \qquad \begin{matrix} R_2 - 3R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 0 & -5 & 19 & 9 \\ 0 & 1 & -5 & -3 \end{bmatrix} \qquad R_3 - 2R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 0 & 1 & -5 & -3 \\ 0 & -5 & 19 & 9 \end{bmatrix} \qquad R_{23}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & -6 & -6 \end{bmatrix} \qquad \begin{matrix} R_1 - 2R_2 \\ R_3 + 5R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad -\frac{1}{6}R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{matrix} R_1 - 2R_3 \\ R_2 + 5R_3 \end{matrix}$$

Since matrix A is non singular, so unique soln. exists

$\therefore$ Soln. is

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 1 \end{array}\right\}$$

Q16
$$x_1 + 3x_2 + 5x_3 - 4x_4 = 1$$
$$x_1 + 2x_2 + x_3 - x_4 + x_5 = -1$$
$$x_1 - 2x_2 + 3x_3 + 2x_4 - x_5 = 3$$
$$x_1 + 5x_2 + 3x_3 + x_4 + x_5 = -11$$
$$x_1 + 3x_2 - x_3 + x_4 + 2x_5 = -3$$

Sol. Given system is

$$x_1 + 3x_2 + 5x_3 - 4x_4 = 1$$
$$x_1 + 2x_2 + x_3 - x_4 + x_5 = -1$$
$$x_1 - 2x_2 + 3x_3 + 2x_4 - x_5 = 3$$
$$x_1 + 5x_2 + 3x_3 + x_4 + x_5 = -11$$
$$x_1 + 3x_2 - x_3 + x_4 + 2x_5 = -3$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 1 & 2 & 1 & -1 & 1 & -1 \\ 1 & -2 & 3 & 2 & -1 & 3 \\ 1 & 5 & 3 & 1 & 1 & -11 \\ 1 & 3 & -1 & 1 & 2 & -3 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & -1 & -4 & 3 & 1 & -2 \\ 0 & -5 & -2 & 6 & -1 & 2 \\ 0 & 2 & -2 & 5 & 1 & -12 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ R_5 - R_1 \end{array}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & -5 & -2 & 6 & -1 & 2 \\ 0 & 2 & -2 & 5 & 1 & -12 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \qquad (-1)R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 18 & -9 & -6 & 12 \\ 0 & 0 & -10 & 11 & 3 & -16 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \qquad \begin{matrix} R_3 + 5R_2 \\ R_4 - 2R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 6 & -3 & -2 & 4 \\ 0 & 0 & -10 & 11 & 3 & -16 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \qquad \frac{1}{3}R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 6 & -3 & -2 & 4 \\ 0 & 0 & 2 & 5 & -1 & -8 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} R_4 + 2R_3 \\ R_5 + R_3 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 6 & -3 & -2 & 4 \\ 0 & 0 & 2 & 5 & -1 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \frac{1}{2}R_5$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 2 & -13 & 0 & 20 \\ 0 & 0 & 2 & 5 & -1 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad R_3 - 2R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 2 & -13 & 0 & 20 \\ 0 & 0 & 0 & 18 & -1 & -28 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad R_4 - R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 1 & -\frac{13}{2} & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 18 & -1 & -28 \end{bmatrix} \quad \begin{matrix} \frac{1}{2}R_3 \\ R_{45} \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -28 \end{bmatrix} \quad \begin{matrix} R_1 + 4R_4 \\ R_2 + 3R_4 \\ R_3 + \frac{13}{2}R_4 \\ R_5 - 18R_4 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & -49 \\ 0 & 1 & 0 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 28 \end{bmatrix} \quad \begin{matrix} R_1 - 5R_3 \\ R_2 - 4R_3 \\ (-1)R_5 \end{matrix}$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 28 \end{array}\right]$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 3 & 5 & 3\\ 0 & 1 & 9/5 & \frac{2}{5}\\ 0 & -7 & -11 & -2\end{bmatrix} \quad -\frac{1}{5}R_2$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & -2/5 & 9/5\\ 0 & 1 & 9/5 & 2/5\\ 0 & 0 & 8/5 & 4/5\end{bmatrix} \quad \begin{matrix}R_1-3R_2\\ R_3+7R_2\end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & -2/5 & 9/5\\ 0 & 1 & 9/5 & \frac{2}{5}\\ 0 & 0 & 1 & 1/2\end{bmatrix} \quad \frac{5}{8}R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc:c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1/2\\ 0 & 0 & 1 & 1/2\end{array}\right] \quad \begin{matrix}R_1+\frac{2}{5}R_3\\ R_2-\frac{9}{5}R_3\end{matrix}$$

Since matrix A is non singular

So unique soln. exists & soln. is

$$x_1 = 2$$

$$x_2 = -1/2$$

$$x_3 = 1/2$$

Q18

$$5x_1 + 4x_3 + 2x_4 = 3$$

$$x_1 - x_2 + 2x_3 + x_4 = 1$$

$$4x_1 + x_2 + 2x_3 = 1$$

$$x_1 + x_2 + x_3 + x_4 = 0$$

Sol. Given system is

$$5x_1 + 4x_3 + 2x_4 = 3$$

$$x_1 - x_2 + 2x_3 + x_4 = 1$$

$$4x_1 + x_2 + 2x_3 = 1$$

$$x_1 + x_2 + x_3 + x_4 = 0$$

Take augmented matrix

$$A_b = \begin{bmatrix} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 5 & 0 & 4 & 2 & 3 \end{bmatrix} \quad R_{14}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & -3 & -2 & -4 & 1 \\ 0 & -5 & -1 & -3 & 3 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 4R_1 \\ R_4 - 5R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & -3 & -2 & -4 & 1 \\ 0 & -5 & -1 & -3 & 3 \end{bmatrix} \quad R_2 - R_3$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 7 & 8 & 1 \\ 0 & 0 & 14 & 17 & 3 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_3 + 3R_2 \\ R_4 + 5R_2 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 8/7 & 1/7 \\ 0 & 0 & 14 & 17 & 3 \end{bmatrix} \quad \frac{1}{7}R_3$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -5/7 & 2/7 \\ 0 & 1 & 0 & 4/7 & -3/7 \\ 0 & 0 & 1 & 8/7 & 1/7 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_1 + 2R_3 \\ R_2 - 3R_3 \\ R_4 - 14R_3 \end{matrix}$$

$$\overset{R}{\sim}\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{array}{l} R_1 + \frac{5}{7}R_4 \\ R_2 - \frac{4}{7}R_4 \\ R_3 - \frac{8}{7}R_4 \end{array}$$

Since matrix A is non singular

So unique soln. exists & soln. is

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = -1 \\ x_3 = -1 \\ x_4 = 1 \end{array}\right\}$$

Q19 Show that the system

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= a \\ 3x_1 + x_2 - 5x_3 &= b \\ -5x_1 - 5x_2 + 21x_3 &= c \end{aligned}$$

is inconsistent if $c \neq 2a - 3b$

Soln. Given system is

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= a \\ 3x_1 + x_2 - 5x_3 &= b \\ -5x_1 - 5x_2 + 21x_3 &= c \end{aligned}$$

Take augmented matrix

$$A_b = \begin{bmatrix} 2 & -1 & 3 & a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim}\begin{bmatrix} -1 & -2 & 8 & a-b \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix} \qquad R_1 - R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & b-a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix} \quad (-1)R_1$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & -5 & 19 & 3a-2b \\ 0 & 5 & -19 & -5a+5b+c \end{bmatrix} \quad \begin{matrix} R_2-3R_1 \\ R_3+5R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & 1 & -19/5 & \frac{2b-3a}{5} \\ 0 & 5 & -19 & -5a+5b+c \end{bmatrix} \quad -\frac{1}{5}R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -2/5 & \frac{a+b}{5} \\ 0 & 1 & -19/5 & \frac{2b-3a}{5} \\ 0 & 0 & 0 & -2a+3b+c \end{bmatrix} \quad \begin{matrix} R_1-2R_2 \\ R_3-5R_2 \end{matrix}$$

The given system is inconsistent if rank $A \neq$ rank A_b

i.e., only possible when $-2a+3b+c \neq 0$ or $c \neq 2a-3b$

So the given system is inconsistent if $c \neq 2a-3b$

Q20 A soap manufacturer decides to spend 600,000 rupees on radio, magazine & T.V. advertising. If he spends as much on T.V. advertising as on magazines & radio together, and the amount spent on magazines & T.V. combined equals five times that spent on radio. What is the amount to be spent on each type of advertising?

Sol:-

Sol. Let x_1, x_2, x_3 be the amounts in rupees spent on radio, magazines & TV advertising resp. then by given Conditions

$$x_1 + x_2 + x_3 = 600,000 \quad \text{——— (1)}$$

$$x_3 = x_1 + x_2$$

$$\text{or } x_1 + x_2 - x_3 = 0 \quad \text{——— (2)}$$

Also $\quad x_2 + x_3 = 5x_1$

$$\text{or } \quad 5x_1 - x_2 - x_3 = 0. \quad \text{——— (3)}$$

Now we will solve eqs. (1), (2) & (3)

$$x_1 + x_2 + x_3 = 600,000$$

$$x_1 + x_2 - x_3 = 0$$

$$5x_1 - x_2 - x_3 = 0$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 1 & 1 & -1 & 0 \\ 5 & -1 & -1 & 0 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations.

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 0 & 0 & -2 & -600,000 \\ 0 & -6 & -6 & -3000000 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 0 & -6 & -6 & -3000000 \\ 0 & 0 & -2 & -600,000 \end{bmatrix} \quad R_{23}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 0 & 1 & 1 & 500,000 \\ 0 & 0 & -2 & -600,000 \end{bmatrix} \quad -\frac{1}{6} R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 100,000 \\ 0 & 1 & 1 & 500,000 \\ 0 & 0 & -2 & -600,000 \end{bmatrix} \qquad R_1 - R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 100,000 \\ 0 & 1 & 1 & 500,000 \\ 0 & 0 & 1 & 300,000 \end{bmatrix} \qquad -\frac{1}{2}R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 100,000 \\ 0 & 1 & 0 & 200,000 \\ 0 & 0 & 1 & 300,000 \end{array}\right] \qquad R_2 - R_3$$

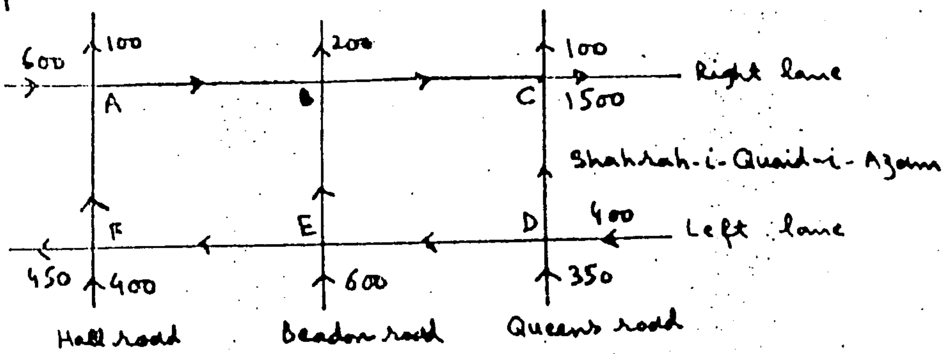
Since matrix A is non singular
So unique soln. exists & soln. is

$x_1 = 100,000$

$x_2 = 200,000$

$x_3 = 300,000$

Q21 Traffic Counters submitted the following information for March 23 from 7 P.M. to 8 P.M. on the following roads of Lahore:



(i) Construct a mathematical model that describes this system, carefully labelling the variables you introduce

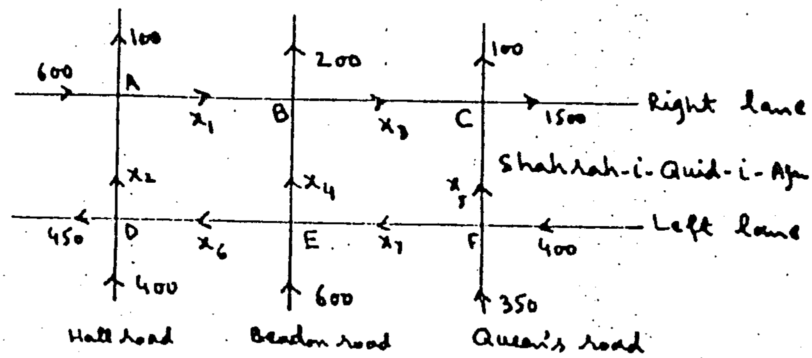
(ii) Show that there must be atleast 50 vehicles travelling on the section of left lane to Hall road from Beadon road during the count.

(iii) The city planners are inclined to take this traffic count as typical rush hour evening traffic in this area. In their planning of the annual closure of left lane between Queen's road & Beadon road for repair, how much traffic can be expected on right lane between Queen's road & Beadon road?

Sol:

(i)

Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be the no. of vehicles along different sections of various roads, as shown in the figure



Equating the incoming flow to the outgoing flow at each junction, we have following mathematical model:

At junction A : $x_2 + 600 = x_1 + 100 \Rightarrow x_1 - x_2 = 500$ —— (1)

At junction B : $x_1 + x_4 = x_3 + 200 \Rightarrow x_1 - x_3 + x_4 = 200$ —— (2)

At junction C : $x_3 + x_5 = 1500 + 100 \Rightarrow x_3 + x_5 = 1600$ —— (3)

At junction D : $x_6 + 400 = x_2 + 450 \Rightarrow x_2 - x_6 = -50$ —— (4)

At junction E : $x_7 + 600 = x_4 + x_6 \Rightarrow x_4 + x_6 - x_7 = 600$ —— (5)

At junction F : $x_5 + x_7 = 400 + 350 \Rightarrow x_5 + x_7 = 750$ —— (6)

So we have the following system of eqs:

$$x_1 - x_2 = 500$$
$$x_1 - x_3 + x_4 = 200$$
$$x_3 + x_5 = 1600$$
$$x_2 - x_6 = -50$$
$$x_4 + x_6 - x_7 = 600$$
$$x_5 + x_7 = 750$$ & its augmented matrix is

$$A_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 750 \end{bmatrix}$$

(ii) From eq. (4) we have

$$x_6 = x_2 + 50$$

which shows that if $x_2 = 0$ then least no. of vehicles travelling on the section of left left lane to Hall road from Beadon road during the Count is 50.

(iii) Because of closure of left lane b/w Queen's road & Beadon road for repair, we have

$$x_7 = 0$$

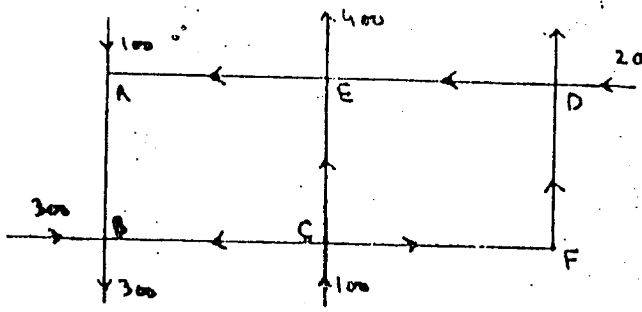
then we can obtain x_3, the no. of vehicles expected on right lane b/w Queen's road & on Beadon road from eqs. (3) & (6)

(6) $\Rightarrow$ $x_5 = 750$ Put in (3)

$$x_3 + 750 = 1600$$

or $x_3 = 1600 - 750 = 850$

Q:22 One part of Lahore's network of traffic is given with the no. of vehicles that enter & leave during a typical rush hour as shown below. All the lanes are one way in the direction indicated by the arrows. 39



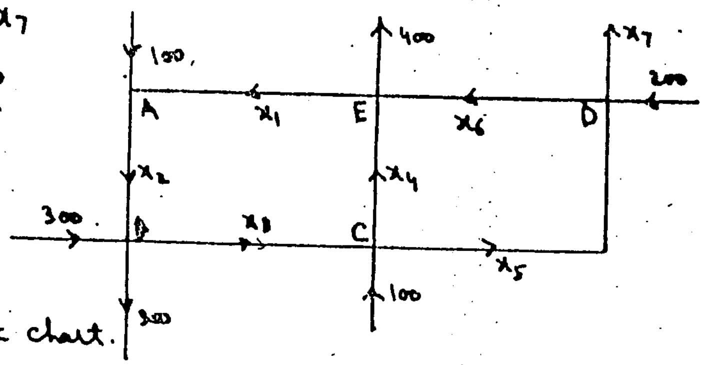
(i) Construct the linear mathematical model that describes this system.

(ii) If the stretch EA is closed for repair, what will be the traffic flow along the other stretches?

(iii) If only 100 vehicles are allowed to pass during the rush hour through EA, how will that effect on other branches?

Sol:-

Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be the no. of vehicles along different sections of various roads in the rush hour as shown by the traffic chart.



Since the no. of incoming & outgoing vehicles at each junction in the network must be equal So the req. mathematical model can be constructed as:

At junction A : $x_1 + 100 = x_2 \Rightarrow x_1 - x_2 = -100$ —— (1)

At junction B : $x_2 + 300 = 300 + x_3 \Rightarrow x_2 - x_3 = 0$ —— (2)

At junction C : $x_3 + 100 = x_4 + x_5 \Rightarrow x_3 - x_4 - x_5 = -100$ —— (3)

At junction D : $x_5 + 200 = x_6 + x_7 \Rightarrow x_5 - x_6 - x_7 = -200$ —— (4)

At junction E : $x_4 + x_6 = 400 + x_1 \Rightarrow x_1 - x_4 - x_6 = -400$ —— (5)

& also $x_7 + 700 = 700 \Rightarrow x_7 = 0$ —— (6)

Now augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -100 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 0 & 0 & -1 & 0 & -1 & -400 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

As rank $A \neq$ rank A_b

So let $x_6 = a$

(ii) When the section EA is closed for repair then $x_1 = 0$

& so from eqs. (1) & (2) $x_2 = 100$ & $x_3 = 100$

From eq. (5) $x_4 = 400 - a$ where $a \leq 400$ —— (7)

From eq. (3) $x_5 = a$ —— (8)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 700$ —— (9)

$\Rightarrow 0 + 100 + 100 + 400 - a + a + a = 700$

or $600 + a = 700 \Rightarrow a = 100$

So $\boxed{x_5 = a = 100}$

& (7) $\Rightarrow x_4 = 400 - a = 400 - 100 = 300$

(iii) Here in this case $x_1 = 100$

(1) $\Rightarrow 100 - x_2 = -100 \Rightarrow x_2 = 200$

(2) $\Rightarrow 200 - x_3 = 0 \Rightarrow x_3 = 200$

(5) $\Rightarrow x_4 + x_6 = 500$ —— (10)

From ③, we have $x_4 + x_5 = 300$ ——— ⑪

$\Rightarrow \quad x_4 + \alpha = 300$

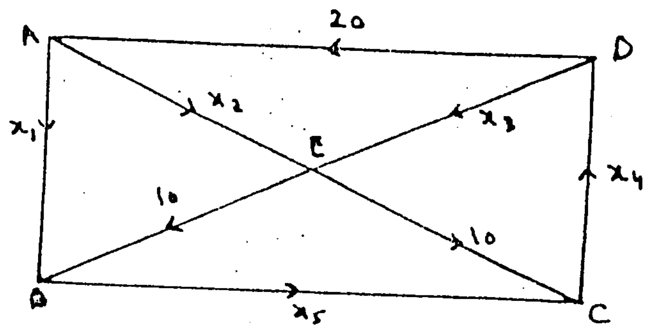
or $x_4 = 300 - \alpha \quad ; \quad 0 \le \alpha \le 300$

from ④ $\quad x_5 + 200 = \alpha \Rightarrow x_5 = \alpha - 200 \; ; \; 0 \le \alpha \le 200$

Hence $0 \le \alpha \le 300$ & $0 \le \alpha \le 200 \Rightarrow 0 \le \alpha \le 200$

Assign arbitrary value to α s.t. $0 \le \alpha \le 200$ we get infinite no. of solutions.

Q23 Set up a system of linear eqs. to represent the network shown in the diagram & solve the system.



If $x_1 = x_3 = 0$, find the flow.

Sol. Here x_1, x_2, x_3, x_4, x_5 be the no. of vehicles along different sections of various roads as shown. Equating the incoming flow to the outgoing flow at each junction, we have following mathematical model.

At junction A $\quad x_1 + x_2 = 20$

At junction B $\quad x_1 + 10 = x_5$

At junction C $\quad x_5 + 10 = x_4$

At junction D $\quad x_3 + 20 = x_4$

At junction E $\quad x_2 + x_3 = 20$

Thus we have the following system

$$x_1 + x_2 = 20$$
$$x_1 - x_5 = -10$$
$$x_4 - x_5 = 10$$
$$x_3 - x_4 = -20$$
$$x_2 + x_3 = 20$$

The augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations.

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & -1 & 0 & 0 & -1 & -30 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{bmatrix} \quad R_2 - R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{bmatrix} \quad (-1)R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_5 - R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{bmatrix} \quad R_{34}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{bmatrix} \quad R_5 - R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 + R_4 \\ R_5 - R_4 \end{matrix}$$

Here rank A = rank Ab

So soln. exists & soln. is

$$x_1 - x_5 = -10$$
$$x_2 + x_5 = 30$$
$$x_3 - x_5 = -10$$
$$x_4 - x_5 = 10$$

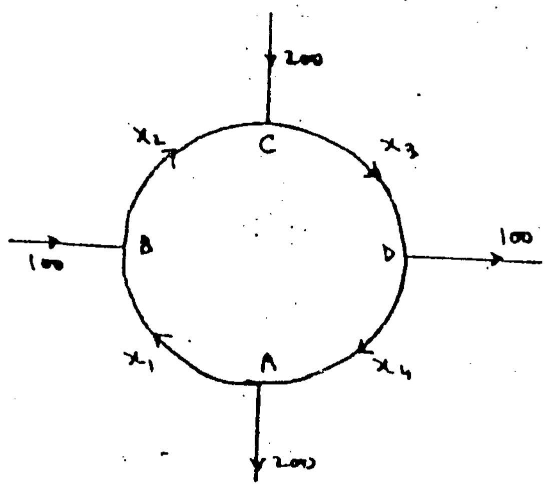
or
$$\left.\begin{aligned} x_1 &= x_5 - 10 \\ x_2 &= 30 - x_5 \\ x_3 &= x_5 - 10 \\ x_4 &= x_5 + 10 \end{aligned}\right\} \text{ where } x_3 \text{ is arbitrary}$$

when $x_1 = x_3 = 0$ then we see that $x_5 = 0$

Hence

$$x_1 = 0,\ x_2 = 20,\ x_3 = 0,\ x_4 = 20,\ x_5 = 10$$

Q2.4 The flow of traffic at the Kalma Chowk on Ferozepur road, Lahore is shown below:



(i) Solve the system

(ii) Find the traffic flow when $x_4 = 300$

Sol. Here x_1, x_2, x_3, x_4 be the no. of vehicles along different sections of various roads as shown. Equating the incoming traffic to the outgoing traffic at each junction, we have the following mathematical model.

At junction A $\quad x_1 + 200 = x_4$

At junction B $\quad x_1 + 100 = x_2$

At junction C $\quad x_2 + 200 = x_3$

At junction D $\quad x_3 = x_4 + 100$

Thus we have the following system.

$$x_1 - x_4 = -200$$
$$x_1 - x_2 = -100$$
$$x_2 - x_3 = -200$$
$$x_3 - x_4 = 100$$

The augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 1 & -1 & 0 & 0 & -100 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad R_2 - R_1$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad (-1) R_2$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & -1 & 1 & -100 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad R_3 - R_2$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad (-1) R_3$$

$$\overset{R}{\sim} \left[\begin{array}{cccc:c} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 - R_3$$

Here rank A = rank A_b

So soln. exists &

soln. is

$$x_1 - x_4 = -200$$
$$x_2 - x_4 = -100$$
$$x_3 - x_4 = 100$$

or
$$\left.\begin{aligned} x_1 &= x_4 - 200 \\ x_2 &= x_4 - 100 \\ x_3 &= x_4 + 100 \end{aligned}\right\} \text{ where } x_4 \text{ is arbitrary}$$

(ii) If $x_4 = 300$ then

$$x_1 = 300 - 200 = 100$$
$$x_2 = 300 - 100 = 200$$
$$x_3 = 300 + 100 = 400$$

End of linear eqs.

thank God