Second Edition

A FIRST COURSE IN
INTEGRAL EQUATIONS
Solutions Manual

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THIS BOOK IS DEDICATED TO

My wife, our son, and our three daughters
for supporting me in all my endeavors
# Contents

Preface ix

## 1 Introductory Concepts 1

1.2 Classification of Linear Integral Equations 1
1.3 Solution of an Integral Equation 2
1.4 Converting Volterra Equation to an ODE 4
1.5 Converting IVP to Volterra Equation 7
1.6 Converting BVP to Fredholm Equation 11
1.7 Taylor Series 13

## 2 Fredholm Integral Equations 15

2.2 Adomian Decomposition Method 15
2.3 The Variational Iteration Method 22
2.4 The Direct Computation Method 25
2.5 Successive Approximations Method 29
2.6 Successive Substitutions Method 33
2.8 Homogeneous Fredholm Equation 35
2.9 Fredholm Integral Equation of the First Kind 39

## 3 Volterra Integral Equations 41

3.2 Adomian Decomposition Method 41
3.3 The Variational Iteration Method 54
3.4 The Series Solution Method 57
3.5 Converting Volterra Equation to IVP 63
3.6 Successive Approximations Method 67
3.7 Successive Substitutions Method 75
3.9 Volterra Equations of the First Kind 79

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4 Fredholm Integro-Differential Equations 85
  4.3 The Direct Computation Method ............................... 85
  4.4 The Adomian Decomposition Method .......................... 90
  4.5 The Variational Iteration Method ............................ 94
  4.6 Converting to Fredholm Integral Equations ................. 96

5 Volterra Integro-Differential Equations 101
  5.3 The Series Solution Method ................................. 101
  5.4 The Adomian Decomposition Method ......................... 103
  5.5 The Variational Iteration Method .......................... 105
  5.6 Converting to Volterra Equations .......................... 107
  5.7 Converting to Initial Value Problems ...................... 110
  5.8 The Volterra Integro-Differential Equations of the First
       Kind .......................................................... 113

6 Singular Integral Equations 117
  6.2 Abel's Problem ............................................... 117
  6.3 Generalized Abel's Problem ................................ 122
  6.4 The Weakly Singular Volterra Equations ..................... 122
  6.5 The Weakly Singular Fredholm Equations .................... 130

7 Nonlinear Fredholm Integral Equations 133
  7.2 Nonlinear Fredholm Integral Equations ...................... 133
    7.2.1 The Direct Computation Method ......................... 133
    7.2.2 The Adomian Decomposition Method ................... 141
    7.2.3 The Variational Iteration Method ................... 148
  7.3 Nonlinear Fredholm Integral Equations of the First
       Kind .......................................................... 149
  7.4 Weakly-Singular Nonlinear Fredholm Integral Equations ... 153

8 Nonlinear Volterra Integral Equations 157
  8.2 Nonlinear Volterra Integral Equations ...................... 157
    8.2.1 The Series Solution Method ......................... 157
    8.2.2 The Adomian Decomposition Method .................... 163
    8.2.3 The Variational Iteration Method .................... 168
  8.3 Nonlinear Volterra Integral Equations of the First Kind ... 170
    8.3.1 The Series Solution Method .......................... 170
    8.3.2 Conversion to a Volterra Equation of the Second
           Kind ..................................................... 172
  8.4 Nonlinear Weakly-Singular Volterra Equation ................ 173
Preface

This Solutions Manual is provided as a supplement to accompany the text A First Course in Integral Equations, Second Edition. It is intended:

• to develop readers’ skills for each linear or nonlinear exercise provided in the textbook;
• to illustrate the use of the newly developed methods along with the traditional methods used in the text;
• to help readers master integral equations concepts so they can solve regular level and challenging exercises.

This Solutions Manual provides completely explained solutions to all exercises of the textbook. The manual can be used by a wide variety of audience in various fields of mathematics, physics, chemistry and engineering. Because the text A First Course in Integral Equations, Second Edition contains the newly developed methods and some of the classical methods, we have aimed to make the Solutions Manual self-explanatory, instructional, and useful. The fully-worked solutions are explained in a systematic way focusing on the needs, suggestions and expectations of the readers. The following distinguishing features make this Solutions Manual significant:

• it provides a fully worked solution for each problem in the text;
• it applies the new methods along with some of the classic methods for most problems using the same strategies, notations, and terminologies used in the text A First Course in Integral Equations, Second Edition;
• it illustrates the solutions in a systematic fashion consistent with the material presented in the text;
• it assists in helping readers gain knowledge and understanding of the mathematical methods as they work through the problems and study the solutions;
• it addresses all questions and suggestions made by students and scholars who used the first edition;
• it complements the material and the worked examples of the textbook.

ix

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The text *A First Course in Integral Equations, Second Edition* presents a variety of valuable methods and applications which cannot be found in any other book. For this reason, and based upon a vast request by readers, this *Solutions Manual* has been made to provide detailed explanations and illustrations for solving each problem of the second edition.

I would like to acknowledge my wife for her genuine patience and I wish to thank her for sincerely encouraging me in all my endeavors, including the writing of the second edition of the textbook as well as the solutions manual. I would also like to acknowledge our son and three daughters for their support and encouragement.

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Chapter 1

Introductory Concepts

1.2 Classification of Linear Integral Equations

Exercises 1.2

1. Fredholm, linear, nonhomogeneous
2. Volterra, linear, nonhomogeneous
3. Volterra, nonlinear, nonhomogeneous
4. Fredholm, linear, homogeneous
5. Fredholm, linear, nonhomogeneous
6. Fredholm, nonlinear, nonhomogeneous
7. Fredholm, nonlinear, nonhomogeneous
8. Fredholm, linear, nonhomogeneous
9. Volterra, nonlinear, nonhomogeneous
10. Volterra, linear, nonhomogeneous
11. Volterra integro-differential equation, nonlinear
12. Fredholm integro-differential equation, linear
13. Volterra integro-differential equation, nonlinear
14. Fredholm integro-differential equation, linear
15. Volterra integro-differential equation, linear

16. \( u(x) = 1 + \int_0^x 4u(t)dt \)

17. \( u(x) = 1 + \int_0^x 3t^2u(t)dt \)

18. \( u(x) = 4 + \int_0^x u^2(t)dt \)
19. \( u'(x) = 1 + \int_0^x 4tu^2(t)dt, \ u(0) = 2 \)

20. \( u'(x) = 1 + \int_0^x 2tu(t)dt, \ u(0) = 0 \)

21. Volterra–Fredholm integral equation, nonlinear, nonhomogeneous

22. Volterra–Fredholm integro-differential equation, linear, nonhomogeneous

23. Volterra–Fredholm integro-differential equation, nonlinear, nonhomogeneous

24. Volterra (singular) integral equation, nonlinear, nonhomogeneous

### 1.3 Solution of an Integral Equation

#### Exercises 1.3

1. Substituting \( u(x) = x + \frac{1}{24} \) in the right hand side yields

\[
RHS = x + \int_0^x \left( t + \frac{1}{24} \right) dt
= x + \left[ \frac{1}{2} t^2 + \frac{1}{24} t \right]_0^1
= x + \frac{1}{24}
= u(x)
= LHS.
\]

2. Substituting \( u(x) = x \) in the right hand side yields

\[
RHS = \frac{2}{3} x + \int_0^1 xt^2 dt
= \frac{2}{3} x + \left[ \frac{1}{4} xt^3 \right]_0^1
= \frac{2}{3} x + \frac{1}{4} x
= \frac{5}{4} x
= u(x)
= LHS.
\]

3. Substituting \( u(x) = 2x \) in the right hand side yields

\[
RHS = x + \int_0^1 4xt^3 dt
= x + \left[ xt^4 \right]_0^1
= 2x
= u(x)
= LHS.
\]

4. Substituting \( u(x) = \sin x \) in the right hand side yields

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1.3. Solution of an Integral Equation

\[ \text{RHS} = x - \int_{0}^{x} (x - t) \sin t \, dt \]
\[ = x - \left[ -x \cos t - \sin t + t \cos t \right]_{0}^{x} \]
\[ = \sin x \]
\[ = u(x) \]
\[ = LHS. \]

5. Substituting \( u(x) = \cosh x \) in the right hand side yields

\[ \text{RHS} = 2 \cosh x - x \sinh x - 1 + \int_{0}^{x} t \cosh t \, dt \]
\[ = 2 \cosh x - x \sinh x - 1 + \left[ t \sinh t - \cosh t \right]_{0}^{x} \]
\[ = \cosh x \]
\[ = u(x) \]
\[ = LHS. \]

6. Substituting \( u(x) = x \) in the right hand side yields

\[ \text{RHS} = x + \frac{1}{5} x^{5} - \int_{0}^{x} t^{4} \, dt \]
\[ = x + \frac{1}{5} x^{5} - \left[ \frac{1}{5} t^{5} \right]_{0}^{x} \]
\[ = x \]
\[ = u(x) \]
\[ = LHS. \]

7. Substituting \( u(x) = x^{2} \) in the right hand side yields

\[ \text{RHS} = 2x - x^{4} + \int_{0}^{x} 4t^{3} \, dt \]
\[ = 2x - x^{4} + \left[ t^{4} \right]_{0}^{x} \]
\[ = 2x \]
\[ = u'(x) \]
\[ = LHS. \]

8. Substituting \( u(x) = \sin x \) in the right hand side yields

\[ \text{RHS} = x \cos x - 2 \sin x + \int_{0}^{x} t \sin t \, dt \]
\[ = x \cos x - 2 \sin x + \left[ \sin t - t \cos t \right]_{0}^{x} \]
\[ = -\sin x \]
\[ = u''(x) \]
\[ = LHS. \]

9. Substituting \( u(x) = 3 \) in the left hand side yields

\[ \text{LHS} = \int_{0}^{x} 3 (x - t)^{2} \, dt \]
\[ = \left[ -(x - t)^{3} \right]_{0}^{x} \]
Chapter 1. Introductory Concepts

\[ x^3 = RHS. \]

10. Substituting \( u(x) = \frac{3}{2} \) in the left hand side yields

\[
LHS = \int_0^x 3 \left( x - t \right)^{\frac{3}{2}} dt
= \left[ \left( x - t \right)^{\frac{5}{2}} \right]_0^x
= x^3
= RHS.
\]

11. Substituting \( u(x) = 2\cos x - 1 \) into both sides yields

\[
2\cos x - 1 = \int_0^x (x - t)(2\cos t - 1) dt
f(x) = x^2
\]

12. Substituting \( u(x) = e^x \) into both sides yields

\[
e^x = f(x) \int_0^x (x - t)(e^t) dt
f(x) = 1 + x
\]

13. Substituting \( u(x) = e^{-x^3} \) into both sides yields

\[
e^{-x^3} = 1 - \alpha \int_0^x 3t^2(e^{-t^3}) dt,
\]

\[ \alpha = 1 \]

14. Substituting \( u(x) = \sin x \) into both sides yields

\[
\sin x = f(x) - 1 \int_0^x t \sin t dt
f(x) = \sin x
\]

1.4 Converting Volterra Equation to an ODE

Exercises 1.4

1. \( \int_0^x 3(x - t)^2u(t)dt \)

2. \( 2xe^{x^3} - e^{x^2} + \int_x^{x^2} te^{xt} dt \)

3. \( \int_0^x 4(x - t)^3u(t)dt \)

4. \( 4\sin 5x - \sin 2x + \int_x^{4x} \cos(x + t) dt \)

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1.4. Converting Volterra Equation to an ODE

5. Differentiating both sides three times, we obtain

\[ u'(x) = 1 + \int_0^x 2(x-t)u(t)dt \]
\[ u''(x) = \int_0^x 2u(t)dt \]
\[ u'''(x) = 2u(x)dt \]

Substituting \( x = 0 \) in the original equation and the first two derivatives, we find \( u(0) = u'(0) = 1, u''(0) = 0 \).

Hence, the equivalent initial value problem is

\[ u'''(x) = 2u(x), \quad u(0) = u'(0) = 1, \quad u''(0) = 0 \]

6. Differentiating both sides two times, we obtain

\[ u'(x) = e^x - \int_0^x u(t)dt \]
\[ u''(x) = e^x - u(x) \]

Substituting \( x = 0 \) in the original equation and the first derivative, we find \( u(0) = u'(0) = 1 \).

Hence, the equivalent initial value problem is

\[ u''(x) + u(x) = e^x, \quad u(0) = u'(0) = 1 \]

7. Differentiating both sides two times, we obtain

\[ u'(x) = 1 + \int_0^x u(t)dt \]
\[ u''(x) = u(x) \]

Substituting \( x = 0 \) in the original equation and the first derivative, we find \( u(0) = 0, \quad u'(0) = 1 \).

Hence, the equivalent initial value problem is

\[ u''(x) - u(x) = 0, \quad u(0) = 0, \quad u'(0) = 1 \]

8. Differentiating both sides two times, we obtain

\[ u'(x) = 1 + \sin x + \int_0^x u(t)dt \]
\[ u''(x) = \cos x + u(x) \]

Substituting \( x = 0 \) in the original equation and the first derivative, we find \( u(0) = -1, \quad u'(0) = 1 \).

Hence, the equivalent initial value problem is

\[ u''(x) - u(x) = \cos x, \quad u(0) = -1, \quad u'(0) = 1 \]

9. Differentiating both sides two times, we obtain

\[ u'(x) = 3 + 10x + u(x) + \int_0^x 2u(t)dt \]
\[ u''(x) = 10 + u'(x) + 2u(x) \]
Chapter 1. Introductory Concepts

Substituting \( x = 0 \) in the original equation and the first derivative, we find
\[
\begin{align*}
    u(0) &= 2, \\
    u'(0) &= 5
\end{align*}
\]
Hence, the equivalent initial value problem is
\[
\begin{align*}
    u''(x) - u'(x) - 2u(x) &= 10, \\
    u(0) &= 2, \\
    u'(0) &= 5
\end{align*}
\]

10. Differentiating both sides two times, we obtain
\[
\begin{align*}
    u'(x) &= 6 + 5u(x) - \int_0^x 6u(t)dt \\
    u''(x) &= 5u'(x) - 6u(x)
\end{align*}
\]
Substituting \( x = 0 \) in the original equation and the first derivative, we find
\[
\begin{align*}
    u(0) &= -5, \\
    u'(0) &= -19
\end{align*}
\]
Hence, the equivalent initial value problem is
\[
\begin{align*}
    u''(x) - 5u'(x) + 6u(x) &= 0, \\
    u(0) &= -5, \\
    u'(0) &= -19
\end{align*}
\]

11. Differentiating both sides we obtain
\[
\begin{align*}
    u'(x) &= \sec^2 x - u(x)
\end{align*}
\]
Substituting \( x = 0 \) in the original equation we find
\[
\begin{align*}
    u(0) &= 0
\end{align*}
\]
Hence, the equivalent initial value problem is
\[
\begin{align*}
    u'(x) + u(x) &= \sec^2 x, \\
    u(0) &= 0
\end{align*}
\]

12. Differentiating both sides three times, we obtain
\[
\begin{align*}
    u'(x) &= 1 + 5x + 3u(x) + \int_0^x \left[ 6 - 5(x - t) \right] u(t)dt \\
    u''(x) &= 5 + 3u'(x) + 6u(x) - 5 \int_0^x u(t)dt \\
    u'''(x) &= 3u''(x) + 6u'(x) - 5u(x)
\end{align*}
\]
Substituting \( x = 0 \) in the original equation and the first two derivatives, we find
\[
\begin{align*}
    u(0) &= 1, \\
    u'(0) &= 4, \\
    u''(0) &= 23
\end{align*}
\]
Hence, the equivalent initial value problem is
\[
\begin{align*}
    u'''(x) - 3u''(x) - 6u'(x) + 5u(x) &= 0, \\
    u(0) &= 1, \\
    u'(0) &= 4, \\
    u''(0) &= 23
\end{align*}
\]

13. \[
\begin{align*}
    u'''(x) - 4u(x) &= 24x, \\
    u(0) &= u'(0) = 0, \\
    u''(0) &= 2
\end{align*}
\]

14. \[
\begin{align*}
    u^{(4)}(x) - u(x) &= 0, \\
    u(0) &= u'(0) = 0, \\
    u''(0) &= 2, \\
    u'''(0) &= 0
\end{align*}
\]
1.5 Converting IVP to Volterra Equation

Exercises 1.5

1. We first set
\[ y'(x) = u(x) \]
Integrating both sides from 0 to \( x \) we find
\[ y(x) - 1 = \int_0^x u(t) \, dt \]
Substituting in the original equation we find
\[ u(x) = -1 - \int_0^x u(t) \, dt \]

2. We first set
\[ y'(x) = u(x) \]
Integrating both sides from 0 to \( x \), we find
\[ y(x) = \int_0^x u(t) \, dt \]
Substituting in the original equation we find
\[ u(x) = x + \int_0^x u(t) \, dt \]

3. We first set
\[ y'(x) = u(x) \]
Integrating both sides from 0 to \( x \) we find
\[ y(x) = \int_0^x u(t) \, dt \]
Substituting in the original equation we find
\[ u(x) = \sec^2(x) - \int_0^x u(t) \, dt \]

4. We first set
\[ y''(x) = u(x) \]
Integrating both sides from 0 to \( x \) twice gives
\[ y'(x) - y'(0) = \int_0^x u(t) \, dt \]
\[ y(x) - y(0) - xy'(0) = \int_0^x \int_0^x u(t) \, dt \, dt \]

or equivalently
\[ y(x) = 1 + \int_0^x (x - t) u(t) \, dt \]
Substituting in the original equation we find
\[ u(x) = -1 - \int_0^x (x - t) u(t) \, dt \]

5. We first set
\[ y''(x) = u(x) \]
Integrating both sides from 0 to \( x \) twice gives
\[ y'(x) - y'(0) = \int_0^x u(t) \, dt \]
Chapter 1. Introductory Concepts

\[ y(x) - y(0) - xy'(0) = \int_0^x \int_0^x u(t) dt \]

or equivalently

\[ y(x) = 1 + x + \int_0^x (x-t)u(t) dt \]

Substituting in the original equation we find

\[ u(x) = 1 + x + \int_0^x (x-t)u(t) dt \]

6. We first set

\[ y''(x) = u(x) \]

Integrating both sides from 0 to \( x \) twice gives

\[ y'(x) - y'(0) = \int_0^x u(t) dt \]

\[ y(x) - y(0) - xy'(0) = \int_0^x \int_0^x u(t) dt \]

or equivalently

\[ y(x) = 1 + x + \int_0^x (x-t)u(t) dt \]

Substituting in the original equation we find

\[ u(x) = -11 - 6x - \int_1^x [5 + 6(x-t)]u(t) dt \]

7. We first set

\[ y''(x) = u(x) \]

Integrating both sides from 0 to \( x \) twice gives

\[ y'(x) - y'(1) = \int_1^x u(t) dt \]

or equivalently

\[ y'(x) = 1 + \int_1^x u(t) dt \]

Substituting in the original equation we find

\[ u(x) = -1 - \int_1^x u(t) dt \]

8. We first set

\[ y''(x) = u(x) \]

Integrating both sides from 0 to \( x \) twice gives

\[ y'(x) - y'(0) = \int_0^x u(t) dt \]

\[ y(x) - y(0) - xy'(0) = \int_0^x \int_0^x u(t) dt \]

or equivalently

\[ y(x) = x + \int_0^x (x-t)u(t) dt \]

Substituting in the original equation we find

\[ u(x) = -1 + 4x + \int_0^x [2(x-t) - 1]u(t) dt \]

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1.5. Converting IVP to Volterra Equation

9. We first set $y''(x) = u(x)$
Integrating both sides from 0 to $x$ twice gives
$$y'(x) - y'(0) = \int_0^x u(t) dt$$
$$y(x) - y(0) - xy'(0) = \int_0^x \int_0^x u(t) dt dt$$
or equivalently
$$y(x) = \int_0^x (x-t)u(t) dt$$
Substituting in the original equation we find
$$u(x) = \sin x - \int_0^x (x-t)u(t) dt$$

10. We first set $y''(x) = u(x)$
Integrating both sides from 0 to $x$ twice gives
$$y'(x) - y'(0) = \int_0^x u(t) dt$$
$$y(x) - y(0) - xy'(0) = \int_0^x \int_0^x u(t) dt dt$$
or equivalently
$$y(x) = 1 - x + \int_0^x (x-t)u(t) dt$$
Substituting in the original equation we find
$$u(x) = x - \sin x + xe^x - e^x - \int_0^x [(x-t)e^x - \sin x] u(t) dt$$

11. We first set $y'''(x) = u(x)$
Integrating both sides from 0 to $x$ three times gives
$$y''(x) - y''(0) = \int_0^x u(t) dt$$
$$y'(x) - y'(0) - xy''(0) = \int_0^x \int_0^x u(t) dt dt$$
$$y(x) - y(0) - xy'(0) - \frac{1}{2} x^2 y''(0) = \int_0^x \int_0^x \int_0^x u(t) dt dt dt$$
or equivalently
$$y(x) = 2 + x^2 + \frac{1}{2} \int_0^x (x-t)^2 u(t) dt$$
Substituting in the original equation we find
$$u(x) = 2x - x^2 + \int_0^x \left[ 1 + (x-t) - \frac{1}{2} (x-t)^2 \right] u(t) dt$$

12. We first set $y'''(x) = u(x)$
Integrating both sides from 0 to $x$ three times gives
Chapter 1. Introductory Concepts

\[ y''(x) - y''(0) = \int_0^x u(t) \, dt \]

\[ y'(x) - y'(0) - xy''(0) = \int_0^x \int_0^x u(t) \, dt \, dt \]

or equivalently

\[ y'(x) = x + \int_0^x (x-t)u(t) \, dt \]

Substituting in the original equation we find

\[ u(x) = -3x - 4 \int_0^x (x-t)u(t) \, dt \]

13. We first set

\[ y^{iv}(x) = u(x) \]

Integrating both sides from 0 to \( x \) three times gives

\[ y''(x) - y''(0) = \int_0^x u(t) \, dt \]

\[ y''(x) - y''(0) - xy'''(0) = \int_0^x \int_0^x \int_0^x u(t) \, dt \, dt \, dt \]

\[ y'(x) - y'(0) - xy''(0) - \frac{1}{2}x^2y'''(0) = \int_0^x \int_0^x \int_0^x \int_0^x u(t) \, dt \, dt \, dt \, dt \]

and by changing multiple integrals to single integrals and by substituting in the original equation we find

\[ u(x) = 2 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \int_0^x \left[ 2(x-t) + \frac{1}{6}(x-t)^3 \right] u(t) \, dt \]

14. We first set

\[ y^{iv}(x) = u(x) \]

Integrating both sides from 0 to \( x \) three times gives

\[ y'''(x) - y'''(0) = \int_0^x u(t) \, dt \]

\[ y'''(x) - y'''(0) - xy''''(0) = \int_0^x \int_0^x \int_0^x \int_0^x u(t) \, dt \, dt \, dt \, dt \]

\[ y''(x) - y''(0) - xy'''(0) - \frac{1}{2}x^2y''''(0) = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x u(t) \, dt \, dt \, dt \, dt \, dt \]

and by changing multiple integrals to single integrals and by substituting in the original equation we find

\[ u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{3!} \int_0^x (x-t)^3 u(t) \, dt \]

15. We first set

\[ y^{iv}(x) = u(x) \]
1.6. Converting BVP to Fredholm Equation

Integrating both sides from 0 to \(x\) three times gives
\[
y'''(x) - y'''(0) = \int_0^x u(t)\,dt
\]
\[
y''(x) - y''(0) - xy'''(0) = \int_0^x \int_0^x u(t)\,dt\,dt
\]
\[
y'(x) - y'(0) - xy''(0) - \frac{1}{2}x^2y'''(0) = \int_0^x \int_0^x \int_0^x u(t)\,dt\,dt\,dt
\]
\[
y(x) - y(0) - xy'(0) - \frac{1}{2}x^2y''(0) - \frac{1}{6}x^3y'''(0) = \int_0^x \int_0^x \int_0^x \int_0^x u(t)\,dt\,dt\,dt\,dt
\]
and by changing multiple integrals to single integrals and by substituting in the original equation we find
\[
u(x) = 2e^x - 1 - x - \int_0^x (x-t)u(t)\,dt
\]

1.6 Converting BVP to Fredholm Equation

Exercises

1. We first set
\[
y''(x) = u(x)
\]
Integrating both sides from 0 to \(x\) twice we find
\[
y'(x) = y'(0) + \int_0^x u(t)\,dt
\]
\[
y(x) = y(0) + xy'(0) + \int_0^x (x-t)u(t)\,dt
\]
Substituting \(x = 1\) in the last equation and using the boundary condition at \(x = 1\) gives
\[
y'(0) = -\int_0^1 (1-t)u(t)\,dt
\]
Accordingly, we obtain
\[
y(x) = -x\int_0^1 (1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt
\]
Substituting in the original equation, we obtain
\[
u(x) = \sin x + 4x\int_0^1 (1-t)u(t)\,dt + 4x\int_0^x (1-t)u(t)\,dt - 4\int_0^x (x-t)u(t)\,dt
\]
This gives
\[
u(x) = \sin x + \int_0^1 K(x,t)u(t)\,dt
\]
\[
K(x,t) = \begin{cases} 4t(1-x) & 0 \leq t \leq x \\ 4x(1-t) & x \leq t \leq 1 \end{cases}
\]

2. We first set
\[
y''(x) = u(x)
\]
Integrating both sides from 0 to \(x\) twice we find
\[
y'(x) = y'(0) + \int_0^x u(t)\,dt
\]
\[
y(x) = y(0) + xy'(0) + \int_0^x (x-t)u(t)\,dt
\]
Substituting \(x = 1\) in the last equation and using the boundary condition
Chapter 1. Introductory Concepts

at $x = 1$ we obtain

$y'(0) = - \int_0^1 (1-t)u(t)dt$

Accordingly, we obtain

$y(x) = -x \int_0^1 (1-t)u(t)dt + \int_0^x (x-t)u(t)dt$

Substituting in the original equation, we obtain

$u(x) = 1 + 2x^2 \int_0^x (1-t)u(t)dt + 2x^2 \int_0^x (1-t)u(t)dt - 2x \int_0^x (x-t)u(t)dt$

This gives

$u(x) = 1 + \int_0^1 K(x,t)u(t)dt$

$K(x,t) = \begin{cases} 
2xt(1-x) & 0 \leq t \leq x \\
2x^2(1-t) & x \leq t \leq 1 
\end{cases}$

3. We first set

$y''(x) = u(x)$

Integrating both sides from 0 to $x$ twice we find

$y'(x) = y'(0) + \int_0^x u(t)dt$

$y(x) = y(0) + \int_0^x xy(0) + \int_0^x (x-t)u(t)dt$

Substituting $x = 1$ in the last equation and using the boundary condition at $x = 1$ we obtain

$y'(0) = -1 - \int_0^1 (1-t)u(t)dt$

Accordingly, we obtain

$y(x) = 1 - x - x \int_0^1 (1-t)u(t)dt + \int_0^x (x-t)u(t)dt$

Substituting in the original equation, we obtain

$u(x) = (2x-1) + x \int_0^x (1-t)u(t)dt + x \int_0^x (1-t)u(t)dt + \int_0^x (x-t)u(t)dt$

This gives

$u(x) = (2x-1) + \int_0^1 K(x,t)u(t)dt$

$K(x,t) = \begin{cases} 
t(1-x) & 0 \leq t \leq x \\
x(1-t) & x \leq t \leq 1 
\end{cases}$

4. We first set

$y''(x) = u(x)$

Integrating both sides from 0 to $x$ twice we find

$y'(x) = y'(0) + \int_0^x u(t)dt$

$y(x) = y(0) + \int_0^x xy(0) + \int_0^x (x-t)u(t)dt$

Substituting $x = 1$ in the first equation and using the boundary condition at $x = 1$ we obtain

$y'(0) = - \int_0^1 (1-t)u(t)dt$

Accordingly, we obtain

$y(x) = 1 - x \int_0^1 u(t)dt + \int_0^x (x-t)u(t)dt$

Substituting in the original equation, we obtain

$u(x) = x - 1 + x \int_0^x u(t)dt + x \int_0^x u(t)dt - \int_0^x (x-t)u(t)dt$
1.7. Taylor Series

This gives \( u(x) = (x - 1) + \int_0^1 K(x, t)u(t)\,dt, \)
\[
K(x, t) = \begin{cases} 
  t & 0 \leq t \leq x \\
  x & x \leq t \leq 1
\end{cases}
\]

1.7 Taylor Series

Exercises 1.7

1. \( f(x) = e^{2x} \)  
2. \( f(x) = e^{-3x} \)
3. \( f(x) = e^x - 1 \)  
4. \( f(x) = \cos(2x) \)
5. \( f(x) = \sin(3x) \)  
6. \( f(x) = \sinh(2x) \)
7. \( f(x) = \cosh(2x) \)  
8. \( f(x) = \cosh(3x) - 1 \)
9. \( f(x) = 1 + \cos(2x) \)  
10. \( f(x) = 1 + \sin x \)
Chapter 2

Fredholm Integral Equations

2.2 Adomian Decomposition Method

Exercises 2.2

1. We set
   \( u_0(x) = \frac{13}{3} x \)
   Hence, we find
   \[ u_1(x) = -\frac{1}{4} \int_0^1 \frac{13}{3} x^2 \, dt \]
   \[ u_1(x) = -\frac{13 x}{3 \times 12} \]
   and
   \[ u_2(x) = \frac{13}{3 \times 144} x \]
   and so on. Accordingly, we find
   \[ u(x) = \frac{13}{4} x \left( 1 - \frac{1}{12} + \frac{1}{144} + \cdots \right) \]
   And by evaluating the infinite geometric series at the right hand side, we find
   \[ u(x) = 4x \]

2. We set
   \( u_0(x) = x^3 - \frac{1}{5} x \)
   Hence, we find
Chapter 2. Fredholm Integral Equations

\[ u_1(x) = \int_0^1 x \left( t^4 - \frac{1}{5} t^2 \right) dt = \frac{2}{15} x \]
and
\[ u_2(x) = \frac{2}{45} x \]
and so on. Accordingly, we find
\[ u(x) = x^3 - \frac{1}{5} x + \frac{2}{15} x \left( 1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) \]
And by evaluating the infinite geometric series at the right hand side, we find
\[ u(x) = x^3 \]

3. We set
\[ u_0(x) = x^2 \]
Hence, we find
\[ u_1(x) = \int_0^1 x t^3 dt = \frac{1}{4} x \]
and
\[ u_2(x) = \frac{1}{12} x \]
and so on. Accordingly, we find
\[ u(x) = x^2 + \frac{1}{4} x \left( 1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) \]
And by evaluating the infinite geometric series at the right hand side, we find
\[ u(x) = x^2 + \frac{3}{8} x \]

4. We set
\[ u_0(x) = e^x \]
Hence, we find
\[ u_1(x) = e^{-1} \int_0^1 e^t dt \]
\[ u_1(x) = 1 - e^{-1} \]
and
\[ u_2(x) = e^{-1} - e^{-2} \]
\[ u_3(x) = e^{-2} - e^{-3} \]
and so on. Accordingly, we find
\[ u(x) = e^x + 1 - \left( e^{-1} - e^{-2} + e^{-2} - e^{-3} + \cdots \right) \]
\[ u(x) = e^x + 1 \]

5. We set
\[ u_0(x) = x + \sin x \]
Hence, we find
2.2. Adomian Decomposition Method

\[ u_1(x) = -x \int_0^{\pi/2} (t + \sin t)\,dt \]
\[ u_1(x) = -(1 + \frac{\pi^2}{8})x \]
and
\[ u_2(x) = (1 + \frac{\pi^2}{8})\frac{\pi^2}{8}x \]
and so on. Accordingly, we find
\[ u(x) = x + \sin x - (1 + \frac{\pi^2}{8}) \left(1 - \frac{\pi^2}{8} + \frac{\pi^4}{64} + \cdots\right) x \]
And by evaluating the infinite geometric series at the right hand side, we find
\[ u(x) = \sin x \]

6. We set
\[ u_0(x) = x + \cos x \]
Hence, we find
\[ u_1(x) = -2x \int_0^{\pi/2} (t + \cos t)\,dt \]
\[ u_1(x) = -(1 + \frac{\pi^2}{36})x \]
and
\[ u_2(x) = (1 + \frac{\pi^2}{36})\frac{\pi^2}{36}x \]
and so on. Accordingly, we find
\[ u(x) = x + \cos x - (1 + \frac{\pi^2}{36}) \left(1 - \frac{\pi^2}{36} + \frac{\pi^4}{36^2} + \cdots\right) x \]
And by evaluating the infinite geometric series at the right hand side, we find
\[ u(x) = \cos x \]

7. We set
\[ u_0(x) = \cos(4x) + \frac{1}{4}x \]
Hence, we find
\[ u_1(x) = -x \int_0^{\pi} \left(\cos 4t + \frac{1}{4}t\right)\,dt \]
\[ u_1(x) = -\frac{1}{4}(1 + \frac{\pi^2}{2 \times 8^2})x \]
and
\[ u_2(x) = \frac{1}{4}(1 + \frac{\pi^2}{2 \times 8^2}) \frac{\pi^2}{2 \times 8^2}x \]
and so on. Accordingly, we find
\[ u(x) = \cos 4x + \frac{1}{4}x - \frac{1}{4} \left(1 + \frac{\pi^2}{2 \times 8^2}\right) \left(1 - \frac{\pi^2}{2 \times 8^2} + \frac{\pi^4}{4 \times 8^4} + \cdots\right) x \]
Hence, we find
\[ u(x) = \cos 4x \]

8. We set
\[ u_0(x) = \sinh x - e^{-x} \]
Hence, we find
\[ u_1(x) = \int_0^1 x t (\sinh t - e^{-t}) dt \]
\[ u_1(x) = 2e^{-1}x \]
and
\[ u_2(x) = \frac{2}{3}e^{-1}x \]
and so on. Accordingly, we find
\[ u(x) = \sinh x - e^{-x} + \frac{2}{3}e^{-x} \left( 1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) \]
And by evaluating the infinite geometric series at the right hand side, we find
\[ u(x) = \sinh x \]

9. We set
\[ u_0(x) = 2e^{2x} + (1 - e^2)x \]
Hence, we find
\[ u_1(x) = x \int_0^1 (2e^{2t} + (1 - e^2)t) dt \]
\[ u_1(x) = -\frac{1}{2}(1 - e^2)x \]
and
\[ u_2(x) = -\frac{1}{4}(1 - e^2)x \]
and so on. Accordingly, we find
\[ u(x) = 2e^{2x} + (1 - e^2)x - \frac{1}{2}(1 - e^2)x \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \]
And by evaluating the infinite geometric series at the right hand side, we find
\[ u(x) = 2e^{2x} \]

10. We set
\[ u_0(x) = 1 + \sec^2 x \]
Hence, we find
\[ u_1(x) = -\int_0^\pi (1 + \sec^2 t) dt \]
\[ u_1(x) = -\left( 1 + \frac{\pi}{4} \right) \]
and
2.2. Adomian Decomposition Method

\[ u_2(x) = (1 + \frac{\pi}{4})^{\frac{\pi}{4}} \]
and so on. Accordingly, we find
\[ u(x) = 1 + \sec^2x - (1 + \frac{\pi}{4}) \left( 1 - \frac{\pi}{4} + \frac{\pi^2}{16} + \cdots \right) \]

And by evaluating the infinite geometric series at the right hand side, we find
\[ u(x) = \sec^2x \]

11. We set
\[ u_0(x) = \sin x \]
Hence, we find
\[ u_1(x) = e^{\sin^{-1}x} \int_{-1}^{1} \sin t dt \]
\[ u_1(x) = 0 \]
and
\[ u_n(x) = 0, \text{ for } n \geq 1 \]
Accordingly, we find
\[ u(x) = \sin x \]

12. We set
\[ u_0(x) = \tan x \]
Hence, we find
\[ u_1(x) = -e^{\tan^{-1}x} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan t dt \]
\[ u_1(x) = 0 \]
and
\[ u_n(x) = 0, \text{ for } n \geq 1 \]
Accordingly, we find
\[ u(x) = \tan x \]

13. We set
\[ u_0(x) = \tan^{-1}x \]
Hence, we find
\[ u_1(x) = \frac{1}{2} \ln 2 - \frac{\pi}{2} x + x \int_{0}^{1} \tan^{-1} dt \]
\[ u_1(x) = \frac{1}{2} \ln 2 - \frac{\pi}{2} x + x[t(tan^{-1}t) - \frac{1}{2} \ln(1 + t^2)]_0^1 \]
and, hence
\[ u_n(x) = 0, \text{ for } n \geq 1 \]
Accordingly, we find
\[ u(x) = \tan^{-1}x \]

14. We set
Chapter 2. Fredholm Integral Equations

\[ u_0(x) = \cosh x \]

Hence, we find
\[ u_1(x) = (\sinh 1)x + (e^{-1} - 1) - \int_0^1 (x - t) \cosh t dt \]

\[ u_1(x) = (\sinh 1)x + (e^{-1} - 1) - [x \sinh t - t \sinh t + \cosh t]_0^1 \]

and, hence
\[ u_0(x) = 0, \text{ for } n \geq 1 \]

Accordingly, we find
\[ u(x) = \cosh x \]

15. We set
\[ u_0(x) = \frac{1}{1 + x^2} \]
Hence, we find
\[ u_1(x) = (e^{\pi} - e^{-\pi})x - x \int_1^0 1 + t^2 e^{-\tan^{-1} t} dt \]

Using integration by substitution, we set \( v = \tan^{-1} t \), we obtain
\[ u_1(x) = (e^{\pi} - e^{-\pi})x - x[e^{-\tan^{-1} t}]_1^0 \]
and, hence
\[ u_0(x) = 0, \text{ for } n \geq 1 \]

Accordingly, we find
\[ u(x) = \frac{1}{1 + x^2} \]

16. We set
\[ u_0(x) = \frac{1}{\sqrt{1 - x^2}} \]
Hence, we find
\[ u_1(x) = (e^{\pi} - 1)x - x \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - t^2}} e^{-\sin^{-1} t} dt \]

Using integration by substitution, we set \( v = \sin^{-1} t \), we obtain \( u_1(x) = (e^{\pi} - 1)x - x[e^{-\sin^{-1} t}]_0^{\frac{\pi}{2}} \]
and, hence
\[ u_0(x) = 0, \text{ for } n \geq 1 \]
accordingly, we find
\[ u(x) = \frac{1}{\sqrt{1 - x^2}} \]

17. We set
\[ u_0(x) = \frac{1}{1 + x^2} \]
Hence, we find
\[ u_1(x) = \frac{\pi^2}{32}x - x \int_0^1 \frac{1}{1 + t^2} \tan^{-1} t dt \]

Using integration by substitution, we set \( v = \tan^{-1} t \), we obtain
2.2. Adomian Decomposition Method

\[ u_1(x) = \frac{\pi^2}{32} x - x\left[\frac{\tan^{-1}t}{2}\right]_0^1 \]

and, hence
\[ u_0(x) = 0, \text{ for } n \geq 1 \]

Accordingly, we find
\[ u(x) = \frac{1}{1 + x^2} \]

18. We set
\[ u_0(x) = \cos^{-1}x \]

Hence, we find
\[ u_1(x) = -\pi x + x\int_{-1}^1 \cos^{-1}tdt \]
\[ u_1(x) = -\pi x + x[t(\cos^{-1}t) - \sqrt{1 - t^2}]_1^{1-} \]

and, hence
\[ u_n(x) = 0, \text{ for } n \geq 1 \]

Accordingly, we find
\[ u(x) = \cos^{-1}x \]

19. We set
\[ u_0(x) = x\tan^{-1}x \]

Hence, we find
\[ u_1(x) = \left(\frac{\pi}{4} - \frac{1}{2}\right)x - x\int_0^1 t\tan^{-1}dt \]
\[ u_1(x) = \left(\frac{\pi}{4} - \frac{1}{2}\right)x - \frac{1}{2}x[(1 + t^2)\tan^{-1}t - t]_0^1 \]

and, hence
\[ u_n(x) = 0, \text{ for } n \geq 1 \]

Accordingly, we find
\[ u(x) = x\tan^{-1}x \]

20. We set
\[ u_0(x) = x\sin^{-1}x + 1 \]

Hence, we find
\[ u_1(x) = -(\frac{\pi}{8} + 1)x + x\int_0^1 (\sin^{-1}t + 1)dt \]
\[ u_1(x) = \left(\frac{\pi}{8} + 1\right)x + \frac{1}{4}x[(2t^2 - 1)\sin^{-1}t + t\sqrt{1 - t^2}]_0^1 \]

and, hence
\[ u_n(x) = 0, \text{ for } n \geq 1 \]

Accordingly, we find
\[ u(x) = x\sin^{-1}x + 1 \]

21. We set
Chapter 2. Fredholm Integral Equations

\begin{align*}
u_0(x) &= \frac{\sin x}{1+\sin x} + x - \frac{\pi}{2}x \\
u_1(x) &= \int_0^\pi xu_0(t) \, dt = -x + \frac{\pi}{2}x + \frac{\pi^2}{8}x - \frac{\pi^3}{16}x
\end{align*}

By cancelling the noise terms, we find
\begin{equation*}
u(x) = \frac{\sin x}{1+\sin x}
\end{equation*}

22. We set
\begin{align*}
u_0(x) &= \frac{\sin x}{1+\cos x} - x \ln 2 \\
u_1(x) &= \int_0^\pi xu_0(t) \, dt = x \ln 2 - \frac{\pi^2}{8}x \ln 2
\end{align*}

By cancelling the noise terms, we find
\begin{equation*}
u(x) = \frac{\sin x}{1+\cos x}
\end{equation*}

23. We set
\begin{align*}
u_0(x) &= \frac{\sec^2 x}{1+\tan x} - x \ln 2 \\
u_1(x) &= \int_0^\pi xu_0(t) \, dt = x \ln 2 - \frac{\pi^2}{8}x \ln 2
\end{align*}

By cancelling the noise terms, we find
\begin{equation*}
u(x) = \frac{\sec^2 x}{1+\tan x}
\end{equation*}

24. We set
\begin{align*}
u_0(x) &= 1 + \sin x - x - \frac{\pi^2}{8}x \\
u_1(x) &= \int_0^\pi xu_0(t) \, dt = x + \frac{\pi^2}{8}x + \cdots
\end{align*}

By cancelling the noise terms, we find
\begin{equation*}
u(x) = 1 + \sin x
\end{equation*}

25. We set
\begin{align*}
u_0(x) &= 1 + \sin x + \cos x - 2x - \frac{\pi}{2}x + \frac{\pi^2}{8}x + \frac{\pi^3}{8} \\
u_1(x) &= \int_0^\pi (x-t)u_0(t) \, dt = 2x + \frac{\pi}{2}x - \frac{\pi^2}{8}x + \cdots
\end{align*}

By cancelling the noise terms, we find
\begin{equation*}
u(x) = 1 + \sin x + \cos x
\end{equation*}

26. We set
\begin{align*}
u_0(x) &= x \sin x - x - 2 + \pi \\
u_1(x) &= \int_0^\pi (x-t)u_0(t) \, dt = x + 2 - \pi + \cdots
\end{align*}

By cancelling the noise terms, we find
\begin{equation*}
u(x) = x \sin x
\end{equation*}

2.3 The Variational Iteration Method

Exercises 2.3

1. Differentiating both sides gives
\begin{align*}
u'(x) &= 3x^2 - \frac{1}{5} + \int_0^1 tu(t) \, dt, \nu(0) = 0 \\
u_0(x) &= 0
\end{align*}
2.3. The Variational Iteration Method

\[ u_1(x) = x^3 - \frac{1}{5}x \]
\[ u_2(x) = x^3 - \frac{1}{15}x \]
\[ u_3(x) = x^3 - \frac{1}{45}x \]
\[ \vdots \]
\[ u_n(x) = x^3 - \frac{1}{5\cdot3^{n-1}}x, \quad n \geq 1 \]
\[ u(x) = x^3 \]

2. Differentiating both sides gives
\[ u'(x) = e^x - 1 + \int_0^1 tu(t) \, dt, \quad u(0) = 1 \]
\[ u_0(x) = 1 \]
\[ u_1(x) = e^x - \frac{1}{2}x \]
\[ u_2(x) = e^x - \frac{1}{5}x \]
\[ u_3(x) = e^x - \frac{1}{15}x \]
\[ \vdots \]
\[ u_n(x) = e^x - \frac{1}{2\cdot3^{n-1}}x, \quad n \geq 1 \]
\[ u(x) = e^x \]

3. Differentiating both sides gives
\[ u'(x) = \frac{2}{3} + \int_0^1 tu(t) \, dt, \quad u(0) = 0 \]
\[ u_0(x) = 0 \]
\[ u_1(x) = \frac{2}{3}x \]
\[ u_2(x) = \frac{8}{9}x \]
\[ u_3(x) = \frac{26}{27}x \]
\[ \vdots \]
\[ u_n(x) = \frac{3^n-1}{3^n}x, \quad n \geq 1 \]
\[ u(x) = x \]

4. Differentiating both sides gives
\[ u'(x) = 4x^3 + 2x - \frac{5}{12} + \int_0^1 tu(t) \, dt, \quad u(0) = 0 \]
\[ u_0(x) = 0 \]
\[ u_1(x) = x^2 + x^4 - \frac{5}{12}x \]
\[ u_2(x) = x^2 + x^4 - \frac{5}{36}x \]
\[ u_3(x) = x^2 + x^4 - \frac{5}{108}x \]
\[ \vdots \]
\[ u_n(x) = x^2 + x^4 - \frac{5}{12\cdot3^{n-1}}x, \quad n \geq 1 \]
\[ u(x) = x^2 + x^4 \]
5. Differentiating both sides gives
\[ u'(x) = e^x + 2 - \frac{3}{4} + \int_0^1 t u(t) \, dt, \quad u(0) = 1 \]
\[ u_0(x) = 1 \]
\[ u_1(x) = e^x + \frac{13}{8} x \]
\[ u_2(x) = e^x + \frac{27}{32} x \]
\[ u_3(x) = e^x + \frac{133}{128} x \]
\[ \vdots \]
\[ u(x) = e^x \]

6. Differentiating both sides gives
\[ u'(x) = -e^{-x} + 2 + \frac{3}{2} + \int_{-1}^0 t u(t) \, dt, \quad u(0) = 1 \]
\[ u_0(x) = 1 \]
\[ u_1(x) = e^{-x} + \frac{5}{4} x \]
\[ u_2(x) = e^{-x} + \frac{9}{8} x \]
\[ u_3(x) = e^{-x} + \frac{17}{16} x \]
\[ \vdots \]
\[ u(x) = e^{-x} \]

7. Differentiating both sides gives
\[ u'(x) = 1 - \frac{1}{6} x + 2 \int_0^1 t u(t) \, dt, \quad u(0) = 1 \]
\[ u_0(x) = 1 \]
\[ u_1(x) = 1 + x + \frac{9}{12} x^2 \]
\[ u_2(x) = 1 + x + \frac{41}{48} x^2 \]
\[ u_3(x) = 1 + x + \frac{185}{192} x^2 \]
\[ \vdots \]
\[ u(x) = 1 + x + x^2 \]

8. Differentiating both sides gives
\[ u'(x) = e^x - 2x + 2x \int_0^1 t u(t) \, dt, \quad u(0) = 1 \]
\[ u_0(x) = 1 \]
\[ u_1(x) = e^x - \frac{1}{2} x^2 \]
\[ u_2(x) = e^x - \frac{1}{4} x^2 \]
\[ u_3(x) = e^x - \frac{1}{32} x^2 \]
\[ \vdots \]
\[ u_n(x) = e^x - \frac{1}{2^{4n-4}} x^2, \quad n \geq 1 \]
\[ u(x) = e^x \]
2.4. The Direct Computation Method

Exercises 2.4

1. We set
\[ u(x) = xe^x + (\alpha - 1)x \]
where
\[ \alpha = \int_0^1 u(t) \, dt \]
Accordingly, we find
\[ \alpha = \int_0^1 (te^t + (\alpha - 1)t) \, dt \]
This gives
\[ \alpha = 1 \]
Substituting for \( \alpha \) in the first equation we find
\[ u(x) = xe^x \]

2. We set
\[ u(x) = x^2 - \frac{25}{12}x + 1 + \alpha x \]
where
\[ \alpha = \int_0^1 tu(t) \, dt \]
Accordingly, we find
\[ \alpha = \int_0^1 t \left( t^2 - \frac{25}{12}t + 1 + \alpha t \right) \, dt \]
This gives
\[ \alpha = \frac{1}{12} \]
Substituting for \( \alpha \) in the first equation we find
\[ u(x) = x^2 - 2x + 1 \]

3. We set
\[ u(x) = x \sin x + (\alpha - 1)x \]
where
\[ \alpha = \int_0^{\pi/2} u(t) \, dt \]
Accordingly, we find
\[ \alpha = \int_0^{\pi/2} (t \sin t + (\alpha - 1)t) \, dt \]
This gives
\[ \alpha = 1 \]
Substituting for \( \alpha \) in the first equation we find
\[ u(x) = x \sin x \]

4. We set
Chapter 2. Fredholm Integral Equations

\[ u(x) = e^{2x} - \frac{1}{4}(e^2 + 1)x + \alpha x \]

where
\[ \alpha = \int_0^1 t u(t) \, dt \]

Accordingly, we find
\[ \alpha = \int_0^1 t \left( e^{2t} - \frac{1}{4}(e^2 + 1)t + \alpha t \right) \, dt \]

This gives
\[ \alpha = \frac{1}{4}(e^2 + 1) \]

Substituting for \( \alpha \) in the first equation we find
\[ u(x) = e^{2x} \]

5. We set
\[ u(x) = \sec^2 x + (\alpha - \frac{\pi}{4}) \]

where
\[ \alpha = \int_0^{\frac{\pi}{4}} u(t) \, dt \]

Accordingly, we find
\[ \alpha = \int_0^{\frac{\pi}{4}} \left( \sec^2 t + (\alpha - \frac{\pi}{4}) \right) \, dt \]

This gives
\[ \alpha = 1 + \frac{\pi}{4} \]

Substituting for \( \alpha \) in the first equation we find
\[ u(x) = 1 + \sec^2 x \]

6. We set
\[ u(x) = \sin 2x + (\alpha - \frac{1}{2})x \]

where
\[ \alpha = \int_0^{\frac{\pi}{4}} u(t) \, dt \]

Accordingly, we find
\[ \alpha = \int_0^{\frac{\pi}{4}} \left( \sin 2t + (\alpha - \frac{1}{2})t \right) \, dt \]

This gives
\[ \alpha = \frac{1}{2} \]

Substituting for \( \alpha \) in the first equation we find
\[ u(x) = \sin 2x \]

7. We set
\[ u(x) = x^2 - \frac{1}{3}x - \frac{1}{4} + \alpha (x + 2) \]

where
2.4. The Direct Computation Method

\[ \alpha = \int_0^1 u(t) \, dt \]

Accordingly, we find

\[ \alpha = \int_0^1 \left( t^2 - \frac{1}{3} t - \frac{1}{4} + \alpha(t + 2) \right) \, dt \]

This gives

\[ \alpha = \frac{1}{18} \]

Substituting for \( \alpha \) in the first equation we find

\[ u(x) = x^2 - \frac{5}{18} x - \frac{5}{36} \]

8. We set

\[ u(x) = \sin x + \cos x + (\alpha - \frac{\pi}{2}) x \]

where

\[ \alpha = \int_0^\pi t u(t) \, dt \]

Accordingly, we find

\[ \alpha = \int_0^\pi t \left( \sin t + \cos t + (\alpha - \frac{\pi}{2}) t \right) \, dt \]

This gives

\[ \alpha = \frac{\pi}{2} \]

Substituting for \( \alpha \) in the first equation we find

\[ u(x) = \sin x + \cos x \]

9. We set

\[ u(x) = \sec x \tan x + (1 - \alpha) x \]

where

\[ \alpha = \int_0^\pi u(t) \, dt \]

Accordingly, we find

\[ \alpha = \int_0^\pi (\sec t \tan t + (1 - \alpha) t) \, dt \]

This gives

\[ \alpha = 1 \]

Substituting for \( \alpha \) in the first equation we find

\[ u(x) = \sec x \tan x \]

10. We set

\[ u(x) = x^2 - \frac{1}{6} x - \frac{1}{24} + \frac{1}{2} \alpha (1 + x) - \frac{1}{2} \beta \]

where

\[ \alpha = \int_0^1 u(t) \, dt \]

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Chapter 2. Fredholm Integral Equations

\[ \beta = \int_0^1 t u(t) \, dt \]

Accordingly, we find

\[ \alpha = \int_0^1 \left( t^2 - \frac{1}{6} t - \frac{1}{24} + \frac{1}{2} \alpha(1 + t) - \frac{1}{2} \beta \right) \, dt \]

\[ \beta = \int_0^1 t \left( t^2 - \frac{1}{6} t - \frac{1}{24} + \frac{1}{2} \alpha(1 + t) - \frac{1}{2} \beta \right) \, dt \]

This gives

\[ \alpha = \frac{1}{3} \]

\[ \beta = \frac{1}{4} \]

Substituting for \( \alpha \) and \( \beta \) in the first equation we find

\( u(x) = x^2 \)

11. We set

\( u(x) = \sin x + \frac{1}{4}(\alpha - 1)x \)

where

\[ \alpha = \int_0^{\frac{\pi}{2}} t u(t) \, dt \]

Accordingly, we find

\[ \alpha = \int_0^{\frac{\pi}{2}} t \left( \sin t + \frac{1}{4}(\alpha - 1)t \right) \, dt \]

This gives

\[ \alpha = 1 \]

Substituting for \( \alpha \) in the first equation we find

\( u(x) = \sin x \)

12. We set

\( u(x) = 1 + \alpha \ln x + \beta \)

where

\[ \alpha = \int_0^1 u(t) \, dt \]

\[ \beta = \int_0^1 \ln t \, u(t) \, dt \]

Accordingly, we find

\[ \alpha = \int_0^1 (1 + \alpha \ln t + \beta) \, dt \]

\[ \beta = \int_0^1 \ln t (1 + \alpha \ln t + \beta) \, dt \]

This gives

\[ \alpha = \frac{1}{2} \]
2.5. **Successive Approximations Method**

\[ \beta = 0 \]

Substituting for \( \alpha \) and \( \beta \) in the first equation we find
\[ u(x) = 1 + \frac{1}{2} \ln x \]

13. \( u(x) = x^3 \)

14. \( u(x) = 1 + \frac{\pi}{4} \sec^2 x \)

### 2.5 Successive Approximations Method

#### Exercises 2.5

1. We select
\[ v_0(x) = 0 \]

Substituting in the original equation we find
\[ u_1(x) = \frac{11}{12} x \]

Substituting \( u_1(x) \) in the original equation we obtain
\[ u_2(x) = \frac{11}{12} x + \frac{1}{4} \int_0^1 \frac{11}{12} xt^2 dt \]

so that
\[ u_2(x) = \frac{12^2 - 1}{12^2} x \]

Proceeding as before we find
\[ u_3(x) = \frac{12^3 - 1}{12^3} x \]

\[ \vdots \]

\[ u_n(x) = \frac{12^n - 1}{12^n} x \]

Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = x \)

2. We select
\[ v_0(x) = 0 \]

Substituting in the original equation we find
\[ u_1(x) = \frac{6}{7} x^3 \]

Substituting \( u_1(x) \) in the original equation we obtain
\[ u_2(x) = \frac{6}{7} x^3 + \frac{5}{7} \int_0^1 \frac{6}{7} x^3 t^4 dt \]

so that
\[ u_2(x) = \frac{7^2 - 1}{7^2} x^3 \]

Proceeding as before we find
Chapter 2. Fredholm Integral Equations

\[ u_3(x) = \frac{7^3 - 1}{7^3} x \]

\[ \vdots \]

\[ u_n(x) = \frac{7^n - 1}{7^n} x^3 \]

Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = x^3 \)

3. We select \( u_0(x) = 0 \)

Substituting in the original equation we find

\[ u_1(x) = \frac{13}{3} x \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = \frac{13}{3} x - \frac{1}{3} \int_0^1 \frac{13}{3} x t^2 dt \]

so that

\[ u_2(x) = \frac{143}{3 \times 12} x \]

Proceeding as before we find

\[ u_3(x) = \frac{1729}{3 \times 12^2} x \]

\[ \vdots \]

\[ u_n(x) = 4x + \left(-1\right)^{n+1} \frac{3 \times 12^{n-1}}{3 \times 12^{n-1}} x, \quad n \geq 1 \]

Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = 4x \)

4. We select \( u_0(x) = 0 \)

Substituting in the original equation we find

\[ u_1(x) = 1 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 + \int_0^1 x dt \]

so that

\[ u_2(x) = 1 + x \]

Proceeding as before we find

\[ u_3(x) = 1 + \frac{3}{2} x \]

\[ u_4(x) = 1 + \frac{3}{4} x \]

\[ \vdots \]

\[ u_n(x) = 1 + \frac{2^n - 2}{2n-1} x, \quad n \geq 1 \]

Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = 1 + 2x \)
2.5. Successive Approximations Method

5. We select
   \( u_0(x) = 0 \)
Substituting in the original equation we find
   \( u_1(x) = \sin x \)
Substituting \( u_1(x) \) in the original equation we obtain
   \[
   u_2(x) = \sin x + \int_0^x \sin x \sin t \cos t \, dt
   \]
so that
   \( u_2(x) = \frac{3}{2} \sin x \)
Proceeding as before we find
   \( u_3(x) = \frac{7}{4} \sin x \)
   
   \[ u_n(x) = \frac{2^n - 1}{2^{n-1}} \sin x, \quad n \geq 1 \]
Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = 2 \sin x \)

6. We select
   \( u_0(x) = 0 \)
Substituting in the original equation we find
   \( u_1(x) = -\frac{1}{2} + \sec^2 x \)
Substituting \( u_1(x) \) in the original equation we obtain
   \[
   u_2(x) = -\frac{1}{2} + \sec^2 x + \frac{1}{2} \int_0^x \left( -\frac{1}{2} + \sec^2 t \right) \, dt
   \]
so that
   \( u_2(x) = -\frac{\pi}{16} + \sec^2 x \)
Proceeding as before we find
   \( u_3(x) = -\frac{\pi^2}{128} + \sec^2 x \)
   
   \[ u_n(x) = -\frac{\pi^{n-1}}{2 \times 8^{n-1}} + \sec^2 x, \quad n \geq 1 \]
Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = \sec^2 x \)

7. We select
   \( u_0(x) = 0 \)
Substituting in the original equation we find
   \( u_1(x) = -\frac{1}{4} + \sec x \tan x \)
\( u_1(x) \) in the original equation we obtain
   \[
   u_2(x) = -\frac{1}{4} + \sec x \tan x + \frac{1}{4} \int_0^x \left( -\frac{1}{4} + \sec t \tan t \right) \, dt
   \]
so that
\[ u_2(x) = -\frac{\pi}{4 \times 12^2} + \sec x \tan x \]
Proceeding as before we find
\[ u_3(x) = -\frac{\pi^2}{4 \times 12^2} + \sec x \tan x \]
\[ \vdots \]
\[ u_n(x) = -\frac{\pi^{n-1}}{4 \times 12^{n-1}} + \sec x \tan x \]
Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = \sec x \tan x \)

8. We select
\[ u_0(x) = 0 \]
Substituting in the original equation we find
\[ u_1(x) = \cosh x + (1 - e^{-1})x \]
Substituting \( u_1(x) \) in the original equation we obtain
\[ u_2(x) = \cosh x + (1 - e^{-1})x - \int_0^1 x t (\cosh t + (1 - e^{-1})t) dt \]
so that
\[ u_2(x) = \cosh x - \frac{1}{3} (1 - e^{-1})x \]
Proceeding as before we find
\[ u_3(x) = \cosh x + \frac{1}{9} (1 - e^{-1})x \]
\[ \vdots \]
\[ u_n(x) = \cosh x + \frac{(-1)^{n+1}}{3^{n-1} (1 - e^{-1})} x, \quad n \geq 1 \]
Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = \cosh x \)

9. We select
\[ u_0(x) = 0 \]
Substituting in the original equation we find
\[ u_1(x) = e^x - (\sinh 1)x \]
Substituting \( u_1(x) \) in the original equation we obtain
\[ u_2(x) = e^x - (\sinh 1)x + \frac{1}{2} \int_{-1}^1 x (e^t - (\sinh 1)t) dt \]
so that
\[ u_2(x) = e^x \]
Proceeding as before we find
\[ u_3(x) = e^x \]
\[ \vdots \]
\[ u_n(x) = e^x \]
Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = e^x \)
2.6. Successive Substitutions Method

10. We select
\[ u_0(x) = 0 \]
Substituting in the original equation we find
\[ u_1(x) = \frac{1}{4}x + \sin x \]
Substituting \( u_1(x) \) in the original equation we obtain
\[ u_2(x) = \frac{1}{4}x + \sin x - \int_0^\frac{\pi}{4} \left( \frac{1}{4}t + \sin t \right) dt \]
so that
\[ u_2(x) = \sin x - \frac{\pi^2}{128}x \]
Proceeding as before we find
\[ u_3(x) = \sin x + \frac{\pi^4}{128 \times 32}x \]
\[ \vdots \]
\[ u_n(x) = \sin x + (-1)^{n+1} \frac{\pi^{2n-2}}{2n+1 \times (4)^{2n-2}}x, \quad n \geq 1 \]
Hence, \( u(x) = \lim_{n \to \infty} u_n(x) = \sin x \)

2.6 Successive Substitutions Method

Exercises 2.6

1. Using the successive substitutions method, and noting that \( f(x) = \frac{1}{6}x, \lambda = \frac{1}{4}, K(x,t) = xt \), we find
\[ u(x) = \frac{11}{6}x + \frac{1}{4} \int_0^1 \frac{11}{6} xt^2 dt + \frac{1}{16} \int_0^1 \frac{11}{6} xt_1^2 t_1^2 dt_1 + \cdots \]
so that
\[ u(x) = \frac{11}{6}x + \frac{11}{72}x + \frac{11}{864}x + \cdots \]
\[ = \frac{1}{6}x [1 + \frac{1}{12} + \frac{1}{144} + \cdots] = 2x \]
by finding the sum of the infinite geometric series.

2. Using the successive substitutions method, and noting that \( f(x) = 1, \lambda = -\frac{1}{4}, K(x,t) = \cos x \), we find
\[ u(x) = 1 - \frac{1}{4} \int_0^\frac{\pi}{4} \cos x dt + \frac{1}{16} \int_0^\frac{\pi}{4} \int_0^\frac{\pi}{4} \cos x \cos t dt_1 dt_2 + \cdots \]
so that
\[ u(x) = 1 - \frac{\pi}{8} \cos x + \frac{\pi}{32} \cos x + \cdots \]
\[ = 1 - \frac{\pi}{8} \cos x [1 + \frac{1}{4} + \frac{1}{16} + \cdots] \]
\[ = 1 - \frac{\pi}{10} \cos x \]
Chapter 2. Fredholm Integral Equations

by finding the sum of the infinite geometric series.

3. Using the successive substitutions method, and noting that 
\[ f(x) = \frac{7}{8}x + 1, \quad \lambda = \frac{1}{2}, \quad K(x, t) = xt, \]
we find
\[ u(x) = \frac{7}{8}x + 1 + \frac{1}{2} \int_0^1 xt(\frac{7}{12}t + 1)dt + \frac{1}{4} \int_0^1 xt^2t_1^2(\frac{7}{12}t_1 + 1)dt_1dt + \cdots \]
so that
\[ u(x) = 1 + \frac{7}{12}x + \frac{25}{72}x \left( 1 + \frac{1}{6} + \frac{1}{36} + \cdots \right) \]
\[ u(x) = 1 + \frac{7}{12}x + \frac{5}{12}x = 1 + x \]

4. Using the successive substitutions method, and noting that 
\[ f(x) = \cos x, \quad \lambda = \frac{1}{2}, \quad K(x, t) = \sin x, \]
we find
\[ u(x) = \cos x + \frac{1}{2} \int_0^{\pi/2} \sin x \cos tdt + \frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin x \sin t \cos t_1 dt_1 dt + \cdots \]
so that
\[ u(x) = \cos x + \frac{1}{2} \sin x + \frac{1}{4} \sin x + \frac{1}{8} \sin x + \cdots \]
\[ = \cos x + \frac{1}{2} \sin x[1 + \frac{1}{2} + \frac{1}{4} + \cdots] = \cos x + \sin x \]

5. Using the successive substitutions method, and noting that 
\[ f(x) = \frac{7}{8}x^2, \quad \lambda = \frac{1}{2}, \quad K(x, t) = x^2t, \]
we find
\[ u(x) = \frac{7}{8}x^2 + \frac{1}{2} \int_0^1 x^2t(\frac{7}{8}t^2)dt + \frac{1}{4} \int_0^1 \frac{7}{8}x^2t^3t_1^3dt_1dt + \cdots \]
so that
\[ u(x) = \frac{7}{8}x^2 + \frac{7}{64}x^2 + \frac{7}{512}x^2 + \cdots \]
\[ u(x) = \frac{7}{8}x^2 \left( 1 + \frac{1}{8} + \frac{1}{64} + \cdots \right) = x^2 \]

6. Using the successive substitutions method, and noting that 
\[ f(x) = \frac{9}{10}x^3, \quad \lambda = \frac{1}{2}, \quad K(x, t) = x^3t, \]
we find
\[ u(x) = \frac{9}{10}x^3 + \frac{1}{2} \int_0^1 x^3t^4dt + \frac{1}{4} \int_0^1 \frac{9}{10}x^3t^4t_1^4dt_1dt + \cdots \]
so that
\[ u(x) = \frac{9}{10}x^3 + \frac{9}{100}x^3 + \frac{9}{1000}x^3 + \cdots \]
\[ u(x) = \frac{9}{10}x^3 \left( 1 + \frac{1}{10} + \frac{1}{100} + \cdots \right) = x^3 \]

7. Using the successive substitutions method, and noting that 
\[ f(x) = \sin x, \quad \lambda = \frac{1}{2}, \quad K(x, t) = \cos x, \]
we find
\[ u(x) = \sin x + \frac{1}{2} \int_0^{\pi/2} \cos x \sin tdt + \frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} \cos x \cos t \sin t_1 dt_1 dt + \cdots \]
2.8. Homogeneous Fredholm Equation

so that
\[ u(x) = \sin x + \frac{1}{2} \cos x + \frac{1}{4} \cos x + \frac{1}{8} \cos x + \cdots \]
\[ = \sin x + \frac{1}{2} \cos x[1 + \frac{1}{2} + \frac{1}{4} + \cdots] = \sin x + \cos x \]

8. Using the successive substitutions method, and noting that
\[ f(x) = 1, \lambda = \frac{1}{2}, K(x,t) = \sin x, \] we find
\[ u(x) = 1 + \frac{1}{2} \int_0^\pi \sin xdt + \frac{1}{4} \int_0^\pi \int_0^\pi \sin x \sin t dt \ dt \ + \cdots \]
so that
\[ u(x) = 1 + \frac{\pi}{4} \sin x + \frac{\pi}{8} \sin x + \frac{\pi}{16} \sin x + \cdots \\
= 1 + \frac{\pi}{4} \sin x[1 + \frac{1}{2} + \frac{1}{4} + \cdots] \\
= 1 + \frac{\pi}{2} \sin x \]

9. Using the successive substitutions method, and noting that
\[ f(x) = 1, \lambda = \frac{1}{2}, K(x,t) = \sec^2 x, \] we find
\[ u(x) = 1 + \frac{1}{2} \int_0^{\pi/4} \sec^2 x dt + \frac{1}{4} \int_0^{\pi/4} \int_0^{\pi/4} \sec^2 x \sec^2 t dt \ dt \ + \cdots \]
so that
\[ u(x) = 1 + \frac{\pi}{8} \sec^2 x + \frac{\pi}{16} \sec^2 x + \cdots \\
= 1 + \frac{\pi}{8} \sec^2 x + \left[1 + \frac{1}{2} + \frac{1}{4} + \cdots\right] \\
= 1 + \frac{\pi}{4} \sec^2 x \]

10. Using the successive substitutions method, and noting that
\[ f(x) = 1, \lambda = \frac{1}{5}, K(x,t) = \sec x \tan x, \] we find
\[ u(x) = 1 + \frac{1}{5} \int_0^{\pi/3} \sec x \tan x dt + \frac{1}{25} \int_0^{\pi/5} \int_0^{\pi/5} \sec x \tan x \sec t \tan t dt \ dt \ + \cdots \]
so that
\[ u(x) = 1 + \frac{\pi}{15} \sec x \tan x + \frac{\pi}{75} \sec x \tan x + \cdots \\
u(x) = 1 + \frac{\pi}{15} \sec x \tan x \left(1 + \frac{1}{5} + \frac{1}{25} + \cdots\right) \\
u(x) = 1 + \frac{\pi}{12} \sec x \tan x \]

2.8 Homogeneous Fredholm Equations

Exercises 2.8

1. Using the direct computation method we find
Chapter 2. Fredholm Integral Equations

\[ u(x) = 2\alpha \lambda \]

where
\[ \alpha = \int_{0}^{1} tu(t)dt \]
Substituting for \( u(t) \) from the above equation we find
\[ \alpha = \int_{0}^{1} 2\alpha \lambda t dt \]
Integrating and solving for \( \lambda \) where \( \alpha \) is a constant we obtain
\[ \lambda = 1 \]
Hence, \( u(x) = 2\alpha \)

2. Using the direct computation method we find
\[ u(x) = 4\alpha \lambda x \]
where
\[ \alpha = \int_{0}^{1} u(t)dt \]
Substituting for \( u(t) \) from the above equation we find
\[ \alpha = \int_{0}^{1} 4\alpha \lambda t dt \]
Integrating and solving for \( \lambda \) where \( \alpha \) is a constant we obtain
\[ \lambda = \frac{1}{2} \]
Hence, \( u(x) = 2\alpha x \)

3. Using the direct computation method we find
\[ u(x) = \alpha \lambda x \]
where
\[ \alpha = \int_{0}^{1} e^t u(t)dt \]
Substituting for \( u(t) \) from the above equation we find
\[ \alpha = \int_{0}^{1} \alpha \lambda t e^t dt \]
Integrating and solving for \( \lambda \) where \( \alpha \) is a constant we obtain
\[ \lambda = 1 \]
Hence, \( u(x) = \alpha x \)

4. Using the direct computation method we find
\[ u(x) = \alpha \lambda \cos x \]
where
\[ \alpha = \int_{0}^{\pi} \sin t u(t)dt \]
Substituting for \( u(t) \) from the above equation we find
\[ \alpha = \int_{0}^{\pi} \alpha \lambda \sin t \cos t dt \]
2.8. Homogeneous Fredholm Equation

Integrating and solving for $\lambda$ where $\alpha$ is a constant we obtain

$\lambda = 2$

$u(x) = 2\alpha \cos x$

5. Expanding $\sin(x + t)$ yields

$u(x) = \frac{2}{\pi} \lambda \int_{0}^{\pi} \sin x \cos t + \cos x \sin t |u(t)|dt$

Using the direct computation method we find

$u(x) = \frac{2}{\pi} \lambda [\alpha \sin x + \beta \cos x]$

where

$\alpha = \int_{0}^{\pi} \cos tu(t)dt$

$\beta = \int_{0}^{\pi} \sin tu(t)dt$

Substituting for $u(t)$ from the above equation we find

$\alpha = \frac{2}{\pi} \lambda \int_{0}^{\pi} \cos t[\alpha \sin t + \beta \cos t]dt$

$\beta = \frac{2}{\pi} \lambda \int_{0}^{\pi} \sin t[\alpha \sin t + \beta \cos t]dt$

Integrating and solving for $\lambda$ where $\alpha$ and $\beta$ are constants we obtain

$\alpha = \lambda \beta$

$\beta = \lambda \alpha$

Hence, we find

$\lambda = \pm 1$

$\beta = \pm \alpha$

Accordingly, $u(x) = \pm \frac{2}{\pi} \alpha (\sin x \pm \cos x)$

6. Expanding $\cos(x - t)$ yields

$u(x) = \frac{2}{\pi} \lambda \int_{0}^{\pi} [\cos x \cos t + \sin x \sin t]u(t)dt$

Using the direct computation method we find

$u(x) = \frac{2}{\pi} \lambda [\alpha \cos x + \beta \sin x]$

where

$\alpha = \int_{0}^{\pi} \cos tu(t)dt$

$\beta = \lambda \int_{0}^{\pi} \sin tu(t)dt$

Substituting for $u(t)$ from the above equation we find

$\alpha = \frac{2}{\pi} \lambda \int_{0}^{\pi} \cos t[\alpha \cos t + \beta \sin t]dt$

$\beta = \frac{2}{\pi} \lambda \int_{0}^{\pi} \sin t[\alpha \cos t + \beta \sin t]dt$
Chapter 2. Fredholm Integral Equations

Integrating and solving for $\lambda$ where $\alpha$ and $\beta$ are constants we obtain

$\alpha = \lambda \alpha$

$\beta = \lambda \beta$

Hence, we find

$\lambda_1 = \lambda_2 = 1$

Accordingly, $u(x) = \frac{2}{\pi}(\alpha \sin x + \beta \cos x)$

7. Using the direct computation method we find $u(x) = \alpha \lambda \sec x$

where

$\alpha = \int_0^\pi \tan tu(t)dt$

Substituting for $u(t)$ from the above equation we find

$\alpha = \int_0^\pi \alpha \lambda \sec t \tan t dt$

Integrating and solving for $\lambda$ where $\alpha$ is a constant we obtain $\lambda = 1$

Hence we find $u(x) = \alpha \sec x$

8. Using the direct computation method we find $u(x) = \alpha \lambda \sec^2 x$

where

$\alpha = \int_0^{\frac{\pi}{4}} u(t)dt$

Substituting for $u(t)$ from the above equation we find

$\alpha = \int_0^{\frac{\pi}{4}} \alpha \lambda \sec^2 t dt$

Integrating and solving for $\lambda$ where $\alpha$ is a constant we obtain $\lambda = 1$

Hence we find $u(x) = \alpha \sec^2 x$

9. Using the direct computation method we find $u(x) = \alpha \lambda \sin^{-1} x$

where

$\alpha = \int_0^1 u(t)dt$

Substituting for $u(t)$ from the above equation we find

$\alpha = \int_0^1 \alpha \lambda \sin^{-1} t dt$

Integrating and solving for $\lambda$ where $\alpha$ is a constant we obtain $\lambda = \frac{2}{\pi - 2}$

Hence we find $u(x) = \frac{2}{\pi - 2} \alpha \sin^{-1} x$

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2.9 Fredholm Integral Equation of the First Kind

10. Using the direct computation method we find
\[ u(x) = \alpha \lambda (3 - \frac{3}{2} x) \]
where
\[ \alpha = \int_{0}^{1} t u(t) dt \]
Substituting for \( u(t) \) from the above equation we find
\[ \alpha = \int_{0}^{1} \alpha \lambda (3t - \frac{3}{2} t^2) dt \]
Integrating and solving for \( \lambda \) where \( \alpha \) is a constant we obtain
\[ \lambda = 1 \]
Hence we find
\[ u(x) = \alpha (3 - \frac{3}{2} x) \]

2.9 Fredholm Integral Equation of the First Kind

Exercises 2.9

1. \( \frac{1}{3} x = \int_{0}^{1} x u(t) dt \)

Using the method of regularization, we obtain
\[ u_\varepsilon(x) = \frac{1}{3\varepsilon} x - \frac{\varepsilon}{7} \int_{0}^{1} t u_\varepsilon(t) dt \]
Using the direct computation method we find
\[ u_\varepsilon(x) = (\frac{1}{3\varepsilon} - \frac{2}{7}) x \]
where
\[ \alpha = \int_{0}^{1} t u_\varepsilon(t) dt \]
This in turn gives
\[ \alpha = \frac{1}{3+\varepsilon^2} \]
\[ u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = x \]

2. \( \frac{2}{3} x = \int_{0}^{1} x t u(t) dt \)

Using the method of regularization, we obtain
\[ u_\varepsilon(x) = \frac{2}{3\varepsilon} x - \frac{\varepsilon}{2} \int_{0}^{1} t^2 u_\varepsilon(t) dt \]
Using the direct computation method we find
\[ u_\varepsilon(x) = (\frac{2}{3\varepsilon} - \frac{2}{7}) x \]
where
\[ \alpha = \int_{0}^{1} t^2 u_\varepsilon(t) dt \]
This in turn gives
\[ \alpha = \frac{1}{4+16\varepsilon^2} \]
\[ u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = 3x \]
Chapter 2. Fredholm Integral Equations

3. \( \frac{1}{2}e^{-x} = \int_{0}^{x} e^{-\frac{t}{2}} u(t) \, dt \)

Using the method of regularization, we obtain

\[ u_{\epsilon}(x) = \frac{1}{2\epsilon} x - \frac{x}{\epsilon} \int_{0}^{\frac{x}{\epsilon}} e^{-\frac{t}{2}} u_{\epsilon}(t) \, dt \]

Using the direct computation method we find

\[ u_{\epsilon}(x) = \left( \frac{1}{2\epsilon} - \frac{1}{2} \right) e^{-x} \]

where
\[ \alpha = \int_{0}^{1} e^{-t} u_{\epsilon}(t) \, dt \]

This in turn gives

\[ \alpha = \frac{1}{2\epsilon + 4\epsilon^2} \]

\[ u(x) = \lim_{\epsilon \to 0} u_{\epsilon}(x) = e^{-x} \]

4. \( \frac{1}{2}x^2 = \int_{-1}^{1} x^2 t^2 u(t) \, dt \)

Using the method of regularization, we obtain

\[ u_{\epsilon}(x) = \frac{1}{2\epsilon} x^2 - \frac{2x}{\epsilon} \int_{-1}^{x/\epsilon} x^2 t^2 u_{\epsilon}(t) \, dt \]

Proceeding as before we find

\[ \alpha = \frac{4}{10 + 2\epsilon} \]

\[ u(x) = \lim_{\epsilon \to 0} u_{\epsilon}(x) = x^2 \]

5. \( \frac{1}{2} \cos x = \int_{0}^{\pi} \cos(x - t) u(t) \, dt \)

Using the method of regularization, we obtain

\[ u_{\epsilon}(x) = \frac{1}{2\epsilon} \cos x - \frac{\alpha}{\epsilon} \cos x - \frac{\beta}{\epsilon} \sin x \]

where
\[ \alpha = \int_{0}^{\pi} \cos(t) u_{\epsilon}(t) \, dt \]
\[ \beta = \int_{0}^{\pi} \sin(t) u_{\epsilon}(t) \, dt \]

This in turn gives

\[ \alpha = \frac{\pi^2}{2\pi + 4\epsilon}, \beta = 0 \]

\[ u(x) = \lim_{\epsilon \to 0} u_{\epsilon}(x) = \cos x \]

6. \( -\frac{1}{2} \cos x = \int_{0}^{\pi} \sin(x - t) u(t) \, dt \)

Using the method of regularization, we obtain

\[ u_{\epsilon}(x) = -\frac{1}{2\epsilon} \cos x - \frac{2\pi}{\epsilon} \sin x + \frac{\beta}{\epsilon} \cos x \]

where
\[ \alpha = \int_{0}^{\pi} \cos(t) u_{\epsilon}(t) \, dt \]
\[ \beta = \int_{0}^{\pi} \sin(t) u_{\epsilon}(t) \, dt \]

This in turn gives

\[ \alpha = -\frac{\pi^2}{\pi^2 + 4\epsilon^2}, \beta = \frac{\pi^3}{2\pi^2 + 8\epsilon^2} \]

\[ u(x) = \lim_{\epsilon \to 0} u_{\epsilon}(x) = \sin x \]
Chapter 3

Volterra Integral Equations

3.2 Adomian Decomposition Method

Exercises 3.2

1. Using the Adomian decomposition method, we set

\[ u_0(x) = 4x + 2x^2 \]

Hence, we find

\[ u_1(x) = -\int_0^x (4t + 2t^2)dt \]
\[ u_1(x) = -2x^2 - \frac{2}{3}x^3 \]

and

\[ u_2(x) = \frac{2}{3}x^3 + \frac{1}{6}x^4 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain
Chapter 3. \textit{Volterra Integral Equations}

\[ u(x) = 4x + 2x^2 - 2x^2 - \frac{2}{3}x^3 + \frac{2}{3}x^3 + \ldots \]
\[ u(x) = 4x \]

2. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 + x - x^2 \]

Hence, we find

\[ u_1(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 \]
\[ u_2(x) = \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{60}x^5 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \ldots \]

Accordingly, we obtain

\[ u(x) = 1 + 2x \]

3. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 \]

Hence, we find

\[ u_1(x) = -\left[ \int_0^x dt \right] \]
\[ u_1(x) = -x \]

and
3.2. Adomian Decomposition Method

\[ u_2(x) = \frac{1}{2!} x^2 \]
\[ u_3(x) = -\frac{1}{3!} x^3 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = 1 - x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \cdots \]
\[ u(x) = e^{-x} \]

4. Using the Adomian decomposition method, we set

\[ u_0(x) = x \]

Hence, we find

\[ u_1(x) = \int_0^x (x - t) dt \]
\[ u_1(x) = \frac{1}{3!} x^3 \]

and

\[ u_2(x) = \frac{1}{5!} x^5 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \]
\[ u(x) = \sinh x \]

5. Using the Adomian decomposition method, we set
Chapter 3. Volterra Integral Equations

\[ u_0(x) = 3x \]

Hence, we find

\[ u_1(x) = -9 \int_0^x 3t(x - t)dt \]
\[ u_1(x) = -\frac{1}{3!}(3x)^3 \]

and

\[ u_2(x) = \frac{1}{5!}(3x)^5 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = 3x - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5 + \cdots \]
\[ u(x) = \sin 3x \]

6. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 \]

Hence, we find

\[ u_1(x) = -4 \int_0^x (x - t)dt \]
\[ u_1(x) = -\frac{1}{2!}(2x)^2 \]

and

\[ u_2(x) = \frac{1}{4!}(2x)^4 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]
3.2. Adomian Decomposition Method

Accordingly, we obtain

\[ u(x) = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 + \cdots \]

\[ u(x) = \cos 2x \]

7. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 + x \]

Hence, we find

\[ u_1(x) = -\int_0^x (1 + t)(x - t)dt \]

\[ u_1(x) = -\frac{1}{2!}x^2 - \frac{1}{3!}x^3 \]

and

\[ u_2(x) = \frac{1}{4!}x^4 + \frac{1}{5!}x^5 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) \]

\[ u(x) = \cos x + \sin x \]

8. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 - x \]

Hence, we find

\[ u_1(x) = -\int_0^x (1 - t)(x - t)dt \]

\[ u_1(x) = -\frac{1}{2!}x^2 + \frac{1}{3!}x^3 \]
and
\[ u_2(x) = \frac{1}{4!}x^4 - \frac{1}{5!}x^5 \]
and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \right) \]

\[ u(x) = \cos x - \sin x \]

9. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 + x \]

Hence, we find

\[ u_1(x) = \int_0^x (1 + t)(x - t)dt \]
\[ u_1(x) = \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \]

and

\[ u_2(x) = \frac{1}{4!}x^4 + \frac{1}{5!}x^5 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \]
\[ u(x) = e^x \]

10. Using the Adomian decomposition method, we set
3.2. Adomian Decomposition Method

\[ u_0(x) = 1 - x \]

Hence, we find

\[ u_1(x) = \int_0^x (1 - t)(x - t)dt \]
\[ u_1(x) = \frac{1}{2!}x^2 - \frac{1}{3!}x^3 \]

and

\[ u_2(x) = \frac{1}{4!}x^4 - \frac{1}{5!}x^5 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots \]
\[ u(x) = e^{-x} \]

11. Using the Adomian decomposition method, we set

\[ u_0(x) = 2 \]

Hence, we find

\[ u_1(x) = \int_0^x 2(x - t)dt \]
\[ u_1(x) = 2 \left( \frac{1}{2!}x^2 \right) \]

and

\[ u_2(x) = 2 \left( \frac{1}{4!}x^4 \right) \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]
Accordingly, we obtain

\[ u(x) = 2 \left( 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \right) \]

\[ u(x) = 2 \cosh x \]

12. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 + x \]

Hence, we find

\[ u_1(x) = \int_0^x (1 + t)dt \]

\[ u_1(x) = x + \frac{1}{2!}x^2 \]

and

\[ u_2(x) = \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \]

\[ u_3(x) = \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = 1 + 2 \left( x + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \right) \]

\[ u(x) = 2e^x - 1 \]

13. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 - \frac{1}{2}x^2 \]

Hence, we find

\[ u_1(x) = -\int_0^x (1 - \frac{1}{2}t^2)(x - t)dt \]
3.2. Adomian Decomposition Method

\[ u_1(x) = -\frac{1}{2!}x^2 + \frac{1}{4!}x^4 \]

and

\[ u_2(x) = +\frac{1}{4!}x^4 + \frac{1}{6!}x^6 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = 1 + 2\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \right) \]

\[ u(x) = 2 \cos x - 1 \]

14. Using the Adomian decomposition method, we set

\[ u_0(x) = 1 + \frac{1}{2}t^2 \]

Hence, we find

\[ u_1(x) = \int_0^x (1 + \frac{1}{2}t^2)(x-t)dt \]

\[ u_1(x) = \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \]

and

\[ u_2(x) = \frac{1}{4!}x^4 + \frac{1}{6!}x^6 \]

and so on. Substitute the components obtained in the decomposition

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

Accordingly, we obtain

\[ u(x) = 1 + 2\left(\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \right) \]

\[ u(x) = 2 \cosh x - 1 \]
Chapter 3. Volterra Integral Equations

15. Using the modified decomposition method we set

\[ u_0(x) = \cos x \]

Hence, we find

\[ u_1(x) = (1 - e^{\sin x})x + x \int_0^x e^{\sin t} \cos t \, dt \]

Using the substitution \( v = \sin t \), we obtain

\[ u_1(x) = 0 \]

and consequently

\[ u_k(x) = 0 \text{ for } k \geq 1 \]

Accordingly, we find

\[ u(x) = \cos x \]

16. Using the modified decomposition method we set

\[ u_0(x) = \sec^2 x \]

Hence, we find

\[ u_1(x) = (1 - e^{\tan x})x + x \int_0^x e^{\tan t} \sec^2 t \, dt \]

Using the substitution \( v = \tan t \), we obtain

\[ u_1(x) = 0 \]

and consequently

\[ u_k(x) = 0 \text{ for } k \geq 1 \]

Accordingly, we find

\[ u(x) = \sec^2 x \]

17. Using the modified decomposition method we set
3.2. Adomian Decomposition Method

\[ u_0(x) = \cosh x \]

Hence, we find

\[ u_1(x) = \frac{x}{2} (1 - e^{\sinh x}) + \frac{x}{2} \int_0^x e^{\sinh t} \cosh t \, dt \]

Using the substitution \( v = \sinh t \), we obtain

\[ u_1(x) = 0 \]

and consequently

\[ u_k(x) = 0 \text{ for } k \geq 1 \]

Accordingly, we find

\[ u(x) = \cosh x \]

18. Using the modified decomposition method we set

\[ u_0(x) = \sinh x \]

Hence, we find

\[ u_1(x) = \frac{1}{10} (e - e^{\cosh x}) + \frac{1}{10} \int_0^x e^{\cosh t} \sinh t \, dt \]

Using the substitution \( v = \cosh t \), we obtain

\[ u_1(x) = 0 \]

and consequently

\[ u_k(x) = 0 \text{ for } k \geq 1 \]

Accordingly, we find

\[ u(x) = \sinh x \]

19. Using the modified decomposition method we set

\[ u_0(x) = x^3 \]
Chapter 3. Volterra Integral Equations

Hence, we find

\[ u_1(x) = -x^5 + 5 \int_0^x t^4 \, dt \]

Using the substitution \( v = \cosh t \), we obtain

\[ u_1(x) = 0 \]

and consequently

\[ u_k(x) = 0 \text{ for } k \geq 1 \]

Accordingly, we find

\[ u(x) = x^3 \]

20. Using the modified decomposition method we set

\[ u_0(x) = \sec x \tan x \]

Hence, we find

\[ u_1(x) = (e - e^{\sec x}) + \int_0^x e^{\sec t} \sec t \tan t \, dt \]

Using the substitution \( v = \sec t \), we obtain

\[ u_1(x) = 0 \]

and consequently

\[ u_k(x) = 0 \text{ for } k \geq 1 \]

Accordingly, we find

\[ u(x) = \sec x \tan x \]

21. Using the noise terms phenomenon we set

\[ u_0(x) = 8x - 4x^3 \]

Hence, we find
3.2. **Adomian Decomposition Method**

\[
u_1(x) = 4x^3 - x^5
\]

By cancelling \(4x^3\), we find

\[
u(x) = 8x
\]

22. Using the **noise terms phenomenon** we set

\[
u_0(x) = 8x^2 - 2x^5
\]

Hence, we find

\[
u_1(x) = 2x^5 - \frac{2}{7}x^8
\]

By cancelling \(2x^5\), we find

\[
u(x) = 8x^2
\]

23. Using the **noise terms phenomenon** we set

\[
u_0(x) = \sec^2 x - \tan x
\]

Hence, we find

\[
u_1(x) = \tan x + \ln(\cos x)
\]

By cancelling \(\tan x\), we find

\[
u(x) = \sec^2 x
\]

24. Using the **noise terms phenomenon** we set

\[
u_0(x) = 1 + x + x^2 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6
\]

Hence, we find

\[
u_1(x) = \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \text{other terms}
\]

By cancelling the noise terms, we find

\[
u(x) = 1 + x + x^2
\]
25. Using the noise terms phenomenon we set

\[ u_0(x) = x \sin x + x \cos x - \sin x \]

Hence, we find

\[ u_1(x) = \sin x - x \cos x + \text{other terms} \]

By cancelling the noise terms, we find

\[ u(x) = x \sin x \]

26. Using the noise terms phenomenon we set

\[ u_0(x) = \cosh^2 x - \frac{1}{4} \sinh(2x) - \frac{1}{2} x \]

Hence, we find

\[ u_1(x) = \frac{1}{4} \sinh(2x) + \frac{1}{2} x + \text{other terms} \]

By cancelling the noise terms, we find

\[ u(x) = \cosh^2 x \]

### 3.3 The Variational Iteration Method

**Exercises 3.3**

1. Differentiating both sides with respect to \( x \) gives

\[ u'(x) = u(x) \]

Using the correction functional gives

\[
\begin{align*}
    u_0(x) &= 1, \\
    u_1(x) &= 1 - \int_0^x \left( u'_0(t) - u_0(t) \right) dt \\
    &= 1 + x, \\
    u_2(x) &= 1 + x - \int_0^x \left( u'_1(t) - u_1(t) \right) dt \\
    &= 1 + x + \frac{1}{2} x^2, \\
    u_3(x) &= 1 + x + \frac{1}{3!} x^2 + \frac{1}{3!} x^3,
\end{align*}
\]

and so on. The solution in a series form is given by
3.3. The Variational Iteration Method

\[ u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \]

that converges to the exact solution 
\[ u(x) = e^x. \]

2. Differentiating both sides with respect to \( x \) gives 
\[ u'(x) = 1 + \int_0^x u(t) \, dt \]
Using the correction functional gives 
\[ u_0(x) = 0, \]
\[ u_1(x) = 0 - \int_0^x \left( u_0'(t) - 1 - \int_0^t u_0(r) \, dr \right) \, dt = x, \]
\[ u_2(x) = x - \int_0^x \left( u_1'(t) - 1 - \int_0^t u_1(r) \, dr \right) \, dt = x + \frac{1}{3!}x^3 \]
\[ u_3(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \]
and so on. The solution in a series form is given by 
\[ u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \]
that converges to the exact solution 
\[ u(x) = \sinh x. \]

3. Differentiating both sides with respect to \( x \) gives 
\[ u'(x) = 3 - 9 \int_0^x u(t) \, dt \]
Using the correction functional gives 
\[ u_0(x) = 0, \]
\[ u_1(x) = 0 - \int_0^x \left( u_0'(t) - 3 + 9 \int_0^t u_0(r) \, dr \right) \, dt = 3x, \]
\[ u_2(x) = 3x - \int_0^x \left( u_1'(t) - 3 + 9 \int_0^t u_1(r) \, dr \right) \, dt = 3x - \frac{1}{3!}(3x)^3 \]
\[ u_3(x) = 3x - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5, \]
and so on. The solution in a series form is given by 
\[ u(x) = (3x)x - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5 + \cdots \]
that converges to the exact solution 
\[ u(x) = \sin(3x). \]

4. Differentiating both sides with respect to \( x \) gives 
\[ u'(x) = -4 \int_0^x u(t) \, dt \]
Using the correction functional gives
Chapter 3. Volterra Integral Equations

\[ u_0(x) = 1, \]
\[ u_1(x) = 1 - \int_0^x \left( u_0(t) + 4 \int_0^t u_0(r) \, dr \right) \, dt = 1 - 2x^2, \]
\[ u_2(x) = 1 - 2x^2 + \int_0^x \left( u_1'(t) + 4 \int_0^t u_0(r) \, dr \right) \, dt = 1 - \frac{1}{2!} (2x)^2 + \frac{1}{4!} (2x)^4 \]

and so on. The solution in a series form is given by
\[ u(x) = \cos(2x). \]

5. Using the correction functional gives
\[ u_0(x) = 1, \]
\[ u_1(x) = 1 + x - \frac{1}{2!} x^2 \]
\[ u_2(x) = 1 + x - \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \]

and so on. The solution in a series form is given by
\[ u(x) = (1 - \frac{1}{2!} (2x)^2 + \frac{1}{3!} (2x)^4 + \cdots) - (x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots) \]

that converges to the exact solution
\[ u(x) = \sin x + \cos x. \]

6. Using the correction functional gives
\[ u_0(x) = 1, \]
\[ u_1(x) = 1 - x - \frac{1}{2!} x^2 \]
\[ u_2(x) = 1 - x - \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \]

and so on. The solution in a series form is given by
\[ u(x) = (1 - \frac{1}{2!} (2x)^2 + \frac{1}{3!} (2x)^4 + \cdots) - (x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots) \]

that converges to the exact solution
\[ u(x) = \cos x - \sin x. \]

7. Differentiating both sides with respect to \( x \) gives
\[ u'(x) = -x + \cos x + \sin x + \int_0^x (x - t)u(t) \, dt \]

Using the correction functional gives
\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u_n'(t) + t - \cos t - \int_0^t (t - r)u_n(r) \, dr \right) \, dt \]

This gives
\[ u_0(x) = 0, \]
\[ u_1(x) = x \]
\[ u_2(x) = x - \frac{1}{2!} x^3 + \frac{1}{4!} x^5 \]

and so on. The solution in a series form is given by
3.4. The Series Solution Method

\[ u(x) = (x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots) \]

that converges to the exact solution \( u(x) = \sin x \).

8. Using the correction functional gives

\[
\begin{align*}
    u_0(x) &= 1, \\
    u_1(x) &= 1 + \frac{1}{2!} x^2 \\
    u_2(x) &= 1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 \\
\end{align*}
\]

and so on. The solution in a series form is given by

\[ u(x) = 1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots \]

that converges to the exact solution \( u(x) = \cosh x \).

3.4 The Series Solution Method

Exercises 3.4

1. Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = 2x + 2x^2 - x^3 + \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right) dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, n \geq 0 \) yields

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = (a_0 + 2)x + (2 + \frac{1}{2!} a_1)x^2 + (\frac{1}{3!} a_2 - 1)x^3 + \cdots \]

Equating the coefficients of like powers of \( x \) in both sides we find

\[
\begin{align*}
    a_0 &= 0 \\
    a_1 &= 2 \\
    a_2 &= 3 \\
    a_k &= 0 \quad \text{for} \quad k \geq 3 \\
\end{align*}
\]

Accordingly, we find

\[ u(x) = 2x + 3x^2 \]

2. Substituting \( u(x) \) by the series
Chapter 3. Volterra Integral Equations

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = 1 + x - \frac{2}{3} x^3 - \frac{1}{2} x^4 + 2 \int_0^x t \left( \sum_{n=0}^{\infty} a_n t^n \right) dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \ n \geq 0 \) yields

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + x + a_0 x^2 + \left( \frac{2}{3} a_1 - \frac{2}{3} \right) x^3 \]

\[ + \left( \frac{1}{2} a_2 - \frac{1}{2} \right) x^4 + \cdots \]

Equating the coefficients of like powers of \( x \) in both sides we find

\[
\begin{align*}
a_0 &= 1 \\
a_1 &= 1 \\
a_2 &= 1 \\
a_k &= 0 \quad \text{for} \quad k \geq 3
\end{align*}
\]

Accordingly we find

\[ u(x) = 1 + x + x^2 \]

3. Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation and using the Taylor expansion for \( \sin x \), lead to

\[ \sum_{n=0}^{\infty} a_n x^n = 1 + 2x - \frac{2}{3!} x^3 - \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right) dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \ n \geq 0 \) yields

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + (2 - a_0) x - \frac{1}{2} a_1 x^2 \]

\[ - \left( \frac{1}{3} a_2 + \frac{2}{3!} \right) x^3 - \frac{1}{4} a_3 x^4 \cdots \]

Equating the coefficients of like powers of \( x \) in both sides we find
3.4. The Series Solution Method

\[ a_0 = 1 \]
\[ a_1 = 1 \]
\[ a_2 = -\frac{1}{2!} \]
\[ a_3 = -\frac{1}{3!} \]
\[ a_4 = \frac{1}{4!} \]

Accordingly, we find
\[ u(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) \]
and in a closed form
\[ u(x) = \cos x + \sin x \]

4. Substituting \( u(x) \) by the series as shown above yields
\[
\sum_{n=0}^{\infty} a_n x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \int_0^x (x - t) \left(\sum_{n=0}^{\infty} a_n t^n\right) \, dt
\]
Evaluating the regular integrals at the right hand side as discussed above gives
\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + x + \left(\frac{1}{2} - \frac{1}{2}a_0\right)x^2
\]
\[
+ \left(\frac{1}{6} - \frac{1}{6}a_1\right)x^3 - \frac{1}{12}a_2 x^4 + \cdots
\]
Equating the coefficients of like powers of \( x \) in both sides we find
\[ a_0 = 1 \]
\[ a_1 = 1 \]
\[ a_2 = 0 \]
\[ a_3 = 0 \]
\[ a_k = 0 \text{ for } k \geq 3 \]

Accordingly, we find
\[ u(x) = 1 + x \]

5. Substituting \( u(x) \) by the series as shown above yields
\[
\sum_{n=0}^{\infty} a_n x^n = -1 - \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) \, dt
\]
Chapter 3. Volterra Integral Equations

Evaluating the regular integrals at the right hand side as discussed above gives

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = -1 - a_0 x - \frac{1}{2} a_1 x^2 - \frac{1}{3} a_2 x^3 - \frac{1}{4} a_3 x^4 - \cdots \]

Equating the coefficients of like powers of \( x \) in both sides we find

- \( a_0 = -1 \)
- \( a_1 = 1 \)
- \( a_2 = \frac{1}{2!} \)
- \( a_3 = \frac{1}{3!} \)
- \( a_4 = -\frac{1}{4!} \)

Accordingly, we find

\[ u(x) = 1 - 1 - (1 - 2x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \cdots) \]

and in a closed form

\[ u(x) = -e^{-x} \]

6. Substituting \( u(x) \) by the series as shown above yields

\[ \sum_{n=0}^{\infty} a_n x^n = 1 - 2 \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right) dt \]

Evaluating the regular integrals at the right hand side as discussed above gives

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 - 2a_0 x - a_1 x^2 - \frac{2}{3} a_2 x^3 - \frac{1}{4} a_3 x^4 - \cdots \]

Equating the coefficients of like powers of \( x \) in both sides we find

- \( a_0 = 1 \)
- \( a_1 = -2 \)
- \( a_2 = 2 \)
- \( a_3 = \frac{4}{3} \)

Accordingly, we find

\[ u(x) = 1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \cdots \]

and in a closed form

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3.4. The Series Solution Method

\( u(x) = e^{-2x} \)

7. Substituting \( u(x) \) by the series as shown above yields

\[
\sum_{n=0}^{\infty} a_n x^n = 1 + x e^x - \int_0^x \left( t \sum_{n=0}^{\infty} a_n t^n \right) dt
\]

Evaluating the regular integrals at the right hand side as discussed above gives

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + x + \left(1 - \frac{1}{2} a_0\right) x^2 + \left(\frac{1}{2!} - \frac{1}{3} a_1\right) x^3 - \cdots
\]

Equating the coefficients of like powers of \( x \) in both sides we find
\[
a_0 = 1 \\
a_1 = 1 \\
a_2 = \frac{1}{2!} \\
a_3 = \frac{1}{3!} \\
\vdots
\]

Accordingly, we find
\[
u(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots
\]

and in a closed form
\[
u(x) = e^x
\]

8. Substituting \( u(x) \) by the series as shown above yields

\[
\sum_{n=0}^{\infty} a_n x^n = x + \int_0^x (x - t) \sum_{n=0}^{\infty} a_n t^n dt
\]

Evaluating the regular integrals at the right hand side as discussed above gives

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{12} a_2 x^4 + \cdots
\]

Equating the coefficients of like powers of \( x \) in both sides we find
\[
a_0 = 0 \\
a_1 = 1 \\
a_2 = 0 \\
a_3 = \frac{1}{3!} \\
\vdots
\]
Accordingly, we find

\[ u(x) = \sinh x \]

9. Substituting \( u(x) \) by the series as shown above yields

\[
\sum_{n=0}^{\infty} a_n x^n = 1 - \frac{1}{2} x^2 - \int_0^x (x-t) \sum_{n=0}^{\infty} a_n t^n \, dt
\]

Evaluating the regular integrals at the right hand side as discussed above gives

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 - \left( \frac{1}{2} \right) a_0 x^2 - \frac{1}{6} a_1 x^3 - \frac{1}{12} a_2 x^4 + \cdots
\]

Equating the coefficients of like powers of \( x \) in both sides we find

\[
a_0 = 1 \\
a_1 = 0 \\
a_2 = -1 \\
a_3 = 0 \\
a_4 = \frac{1}{12}
\]

\vdots

Accordingly, we find

\[ u(x) = 2 \left( 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \cdots \right) - 1 \]

and in a closed form

\[ u(x) = 2 \cos x - 1 \]

10. Substituting \( u(x) \) by the series as shown above yields

\[
\sum_{n=0}^{\infty} a_n x^n = 1 - x - \int_0^x (x-t) \sum_{n=0}^{\infty} a_n t^n \, dt
\]

Evaluating the regular integrals at the right hand side as discussed above gives

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 - x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 - \frac{1}{12} a_2 x^4 + \cdots
\]

Equating the coefficients of like powers of \( x \) in both sides we find

\[
a_0 = 1 \\
a_1 = -1
\]
3.5. Converting Volterra Equation to IVP

\[ a_2 = -\frac{1}{2!} \]
\[ a_3 = \frac{1}{3!} \]
\[ a_4 = \frac{1}{4!} \]
\[ \vdots \]

Accordingly, we find

\[ u(x) = (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots) - \left( x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \right) \]

and in a closed form

\[ u(x) = \cos x - \sin x \]

11. We find \( u(x) = \sinh x \)
12. We find \( u(x) = \sin x \)

### 3.5 Converting Volterra Equation to IVP

**Exercises 3.5**

1. Differentiate both sides using Leibniz rule we obtain

\[ u'(x) = -3u(x) \]

The equivalent initial value problem is then

\[ u'(x) + 3u(x) = 0, \quad u(0) = 1 \]

Solving this equation and using the initial condition, we find

\[ u(x) = e^{-3x} \]

2. Differentiate both sides twice using Leibniz rule we obtain

\[ u'(x) = \int_{0}^{x} u(t) \, dt \]
\[ u''(x) = u(x) \]

The equivalent initial value problem is then
Chapter 3. Volterra Integral Equations

\[ u''(x) - u(x) = 0, \quad u(0) = 1, \quad u'(0) = 0 \]

The characteristic equation is

\[ r^2 - 1 = 0 \]

Solving this equation and using the initial conditions, we find

\[ u(x) = \cosh x \]

3. Differentiate both sides twice using Leibniz rule we obtain

\[ u'(x) = -1 - \int_0^x u(t) \, dt \]
\[ u''(x) = -u(x) \]

The equivalent initial value problem is then

\[ u''(x) + u(x) = 0, \quad u(0) = 1, \quad u'(0) = -1 \]

The characteristic equation is

\[ r^2 + 1 = 0 \]

Solving this equation and using the initial conditions, we find

\[ u(x) = \cos x - \sin x \]

4. Differentiate both sides using Leibniz rule we obtain

\[ u'(x) = 1 + u(x) \]

The equivalent initial value problem is then

\[ u'(x) - u(x) = 1, \quad u(0) = 0 \]

Solving this equation and using the initial conditions, we find

\[ u(x) = e^x - 1 \]

5. Differentiate both sides twice using Leibniz rule we obtain

\[ u'(x) = 1 + \int_0^x u(t) \, dt \]
3.5. Converting Volterra Equation to IVP

\[ u''(x) = u(x) \]

The equivalent initial value problem is then

\[ u''(x) - u(x) = 0, \quad u(0) = 1, \quad u'(0) = 1 \]

The characteristic equation is

\[ r^2 - 1 = 0 \]

Solving this equation and using the initial conditions, we find

\[ u(x) = e^x \]

6. Differentiate both sides four times using Leibniz rule we obtain

\[
\begin{align*}
  u' &= \frac{1}{2} \int_0^x (x-t)^2 u(t) \, dt \\
  u'' &= \int_0^x (x-t)u(t)dt \\
  u''' &= \int_0^x u(t)dt \\
  u^iv &= u(x)
\end{align*}
\]

The equivalent initial value problem is then

\[ u^iv(x) - u(x) = 0, \quad u(0) = 1, \quad u'(0) = u''(0) = u'''(0) = 0 \]

The characteristic equation is

\[ r^4 - 1 = 0 \quad \text{with roots given by} \quad r = \pm 1, \pm i \]

Accordingly we find

\[ u(x) = A \cos x + B \sin x + C \cosh x + D \sinh x \]

Using the initial conditions yields

\[ u(x) = \frac{1}{2} (\cos x + \cosh x) \]

7. Differentiate both sides four times using Leibniz rule we obtain

\[
\begin{align*}
  u' &= 1 + \frac{1}{2} \int_0^x (x-t)^2 u(t) \, dt \\
  u'' &= \int_0^x (x-t)u(t)dt \\
  u''' &= \int_0^x u(t)dt \\
  u^iv &= u(x)
\end{align*}
\]
Chapter 3. Volterra Integral Equations

\[ u''(x) = u(x) \]

The equivalent initial value problem is then

\[ u''(x) - u(x) = 0, \quad u(0) = u''(0) = u'''(0) = 0, \quad u'(0) = 1 \]

The characteristic equation is

\[ r^4 - 1 = 0 \quad \text{with roots given by } r = \pm 1, \pm i \]

Accordingly we find

\[ u(x) = A \cos x + B \sin x + C \cosh x + D \sinh x \]

Using the initial conditions yields

\[ u(x) = \frac{1}{2}(\sin x + \sinh x) \]

8. Differentiate both sides twice using Leibniz rule we obtain

\[ u'(x) = 2x + \int_0^x u(t) \, dt \]

\[ u''(x) = 2 + u(x) \]

The equivalent initial value problem is then

\[ u''(x) - u(x) = 2, \quad u(0) = 0, \quad u'(0) = 0 \]

The characteristic equation of the homogeneous part is

\[ r^2 - 1 = 0 \]

This gives \( u_c(x) = A \cosh x + B \sinh x \)

The particular solution is obtained by substituting \( u_p(x) = C \) where we obtain \( u_p(x) = -2 \)

Using the initial conditions in \( u(x) = u_c(x) + u_p(x) \), we find

\[ u(x) = 2 \cosh x - 2 \]

9. Differentiate both sides twice using Leibniz rule we obtain

\[ u'(x) = 1 + \frac{1}{2} x^2 - \int_0^x u(t) \, dt \]

\[ u''(x) = x - u(x) \]

The equivalent initial value problem is then

\[ u''(x) + u(x) = x, \quad u(0) = 0, \quad u'(0) = 1 \]
3.6. Successive Approximations Method

The characteristic equation of the homogeneous part is

$$r^2 + 1 = 0$$

This gives \( u_c(x) = A \cos x + B \sin x \)

The particular solution is obtained by substituting

$$u_p(x) = C + Dx \ , \text{ where we obtain } u_p(x) = x$$

Using the initial conditions in \( u(x) = u_c(x) + u_p(x) \), we find

$$u(x) = x$$

10. Differentiate both sides twice using Leibniz rule we obtain

\[
\begin{align*}
  u'(x) &= 1 - 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \int_0^x u(t) \, dt \\
  u''(x) &= -2 + x - x^2 - u(x)
\end{align*}
\]

The equivalent initial value problem is then

$$u''(x) + u(x) = -2 + x - x^2, \quad u(0) = 0, \quad u'(0) = 1$$

The characteristic equation of the homogeneous part is

$$r^2 + 1 = 0$$

This gives \( u_c(x) = A \cos x + B \sin x \)

The particular solution is obtained by substituting

$$u_p(x) = C + Dx + Ex^2, \text{ where we obtain } u_p(x) = x - x^2$$

Using the initial conditions in \( u(x) = u_c(x) + u_p(x) \), we find

$$u(x) = x - x^2$$

3.6 Successive Approximations Method

Exercises

1. We select
Chapter 3. Volterra Integral Equations

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 - \int_0^x dt \]

so that

\[ u_2(x) = 1 - x \]

Proceeding as before we find

\[ u_3(x) = 1 - x + \frac{1}{2!}x^2 \]
\[ u_4(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 \]

\[ \vdots \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = e^{-x} \]

2. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 - 9 \int_0^x (x - t) dt \]

so that
3.6. Successive Approximations Method

\[ u_2(x) = 1 - \frac{(3x)^2}{2!} \]

Proceeding as before we find

\[ u_3(x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} \]

\[ \vdots \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = \cos(3x) \]

3. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 + 2x \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 + 2x + 4 \int_0^x (x - t)(1 + 2t)dt \]

so that

\[ u_2(x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} \]

\[ \vdots \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = e^{2x} \]

4. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find...
Chapter 3. Volterra Integral Equations

\[ u_1(x) = 1 - \frac{1}{4}x \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 - \frac{1}{4}x + \frac{1}{16} \int_0^x (x-t)(1 - \frac{1}{4}t)dt \]

so that

\[ u_2(x) = 1 - \frac{1}{4}x + \frac{\left(\frac{1}{4}x\right)^2}{2!} - \frac{\left(\frac{1}{4}x\right)^3}{3!} \]

\[ \vdots \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = e^{-\frac{1}{4}x} \]

5. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 2 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 2 - 2 \int_0^x (x-t)dt \]

so that

\[ u_2(x) = 2\left(1 - \frac{1}{2!}x^2\right) \]

\[ u_3(x) = 2\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4\right) \]

\[ \vdots \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = 2 \cos x \]
3.6. Successive Approximations Method

6. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 - 2 \int_0^x t \, dt \]

so that

\[ u_2(x) = 1 - x^2 \]

\[ u_3(x) = 1 - x^2 + \frac{1}{2!}x^4 \]

\[ \vdots \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = e^{-x^2} \]

7. We select

\[ u_0(x) = x \]

Substituting in the original equation, we find

\[ u_1(x) = x + \frac{1}{3!}x^3 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = x + \int_0^x (x - t)(t + \frac{t^3}{3!}) \, dt \]

so that

\[ u_2(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \]
Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = \sinh x \]

8. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 - \int_0^x (x - t) dt \]

so that

\[ u_2(x) = 1 - \frac{1}{2!} x^2 \]

\[ u_3(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 \]

\[ \vdots \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = \cos x \]

9. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 + x \]

Substituting \( u_1(x) \) in the original equation we obtain
3.6. Successive Approximations Method

\[ u_2(x) = 1 + x - \int_0^x (x - t)(1 + t)dt \]

so that

\[ u_2(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 \]
\[ u_3(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 \]
\[ u_3(x) = \left(1 - \frac{1}{3!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = \cos x + \sin x \]

10. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 - x \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 - x - \int_0^x (x - t)(1 - t)dt \]

so that

\[ u_2(x) = 1 - x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \]
\[ u_3(x) = 1 - x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 \]
\[ u_3(x) = \left(1 - \frac{1}{3!}x^2 + \frac{1}{4!}x^4 - \cdots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) \]

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = \cos x - \sin x \]

11. We select
Chapter 3. Volterra Integral Equations

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 2 - x \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 2 - x + \int_0^x (2 - t) dt \]

so that

\[ u_2(x) = 2 + x - \frac{1}{2!} x^2 \]

\[ u_3(x) = 2 + x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 \]

\[ u_4(x) = 2 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 - \frac{1}{4!} x^4 \]

Accordingly, \( u(x) = \lim_{n \to \infty} u_n(x) = 1 + e^x \)

12. We select

\[ u_0(x) = 0 \]

Substituting in the original equation, we find

\[ u_1(x) = 1 - x - \frac{1}{2!} x^2 \]

Substituting \( u_1(x) \) in the original equation we obtain

\[ u_2(x) = 1 - x - \frac{1}{2!} x^2 + \int_0^x (x - t)(1 - t - \frac{1}{2!} t^2) dt \]

so that

\[ u_2(x) = 1 - \left( x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 \right) \]

\[ u_3(x) = 1 - \left( x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \right) \]
3.7. Successive Substitutions Method

Accordingly,

\[ u(x) = \lim_{n \to \infty} u_n(x) = 1 - \sinh x \]

3.7 Successive Substitutions Method

Exercises 3.7

1. Substituting \( \lambda = 1, f(x) = x, \) and \( K(x, t) = 1 \) into the formula yields

\[ u(x) = x + \int_0^x t\,dt + \int_0^x \int_0^t t\,dt_1\,dt + \cdots \]

Accordingly we obtain \( u(x) \) in a series form

\[ u(x) = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \]

and in a closed form, \( u(x) \) is given by

\[ u(x) = e^x - 1 \]

2. Substituting \( \lambda = 1, f(x) = \frac{1}{2!}x^2, \) and \( K(x, t) = 1 \) into the formula yields

\[ u(x) = \frac{1}{2!}x^2 + \int_0^x \frac{1}{2!}t^2\,dt + \int_0^x \int_0^t \frac{1}{2!}t_1^2\,dt_1\,dt + \cdots \]

Accordingly we obtain \( u(x) \) in a series form

\[ u(x) = \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \]

and in a closed form, \( u(x) \) is given by

\[ u(x) = e^x - x - 1 \]

3. Substituting \( \lambda = -1, f(x) = \frac{1}{3!}x^3, \) and \( K(x, t) = (x-t) \) into the formula yields

\[ u(x) = \frac{1}{3!}x^3 - \int_0^x (x-t) \frac{1}{3!}t^3\,dt + \int_0^x \int_0^t (x-t)(t-t_1) \frac{1}{3!}t_1^3\,dt_1\,dt + \cdots \]

Accordingly we obtain \( u(x) \) in a series form

\[ u(x) = \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots \]

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Chapter 3. Volterra Integral Equations

and in a closed form, \( u(x) \) is given by

\[
u(x) = x - \sin x
\]

4. Substituting \( \lambda = 1 \), \( f(x) = \frac{1}{3!} x^3 \), and \( K(x, t) = (x - t) \) into the formula yields

\[
u(x) = \frac{1}{3!} x^3 + \int_0^x (x-t) \frac{1}{3!} t^3 dt + \int_0^1 \int_0^t (x-t)(t-t_1) \frac{1}{3!} t_1^3 dt_1 dt + \cdots
\]

Accordingly we obtain \( u(x) \) in a series form

\[
u(x) = \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \frac{1}{7!} x^7 + \cdots
\]

and in a closed form, \( u(x) \) is given by

\[
u(x) = -x + \sinh x
\]

5. Substituting \( \lambda = -1 \), \( f(x) = \frac{1}{2!} x^2 \), and \( K(x, t) = (x - t) \) into the formula yields

\[
u(x) = \frac{1}{2!} x^2 - \int_0^x (x-t) \frac{1}{2!} t^2 dt + \int_0^1 \int_0^t (x-t)(t-t_1) \frac{1}{2!} t_1^2 dt_1 dt + \cdots
\]

Accordingly we obtain \( u(x) \) in a series form

\[
u(x) = \frac{1}{2!} x^2 - \frac{1}{4!} x^4 + \frac{1}{6!} x^6 + \cdots
\]

and in a closed form, \( u(x) \) is given by

\[
u(x) = -1 - \cos x
\]

6. Substituting \( \lambda = -1 \), \( f(x) = 1 - \frac{1}{2!} x^2 \), and \( K(x, t) = 1 \) into the formula yields

\[
u(x) = 1 - \frac{1}{2!} x^2 - \int_0^x (1 - \frac{1}{2!} t^2) dt + \int_0^1 \int_0^t (1 - \frac{1}{2!} t_1^2) dt_1 dt + \cdots
\]

Accordingly we obtain

\[
u(x) = 1 - x + \left( \frac{1}{2!} x^2 - \frac{1}{2!} x^2 \right) + \left( \frac{1}{3!} x^3 - \frac{1}{3!} x^3 \right) + \cdots
\]

Hence, \( u(x) \) is given by

\[
u(x) = 1 - x
\]
3.7. **Successive Substitutions Method**

7. Substituting \( \lambda = 2, f(x) = 1, \) and \( K(x, t) = 1 \) into the formula yields

\[
u(x) = 1 + 2\int_0^x dt + \int_0^x \int_0^t dt_1 dt + \cdots
\]

Accordingly we obtain \( u(x) \) in a series form

\[
u(x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots
\]

and in a closed form, \( u(x) \) is given by

\[
u(x) = e^{2x}
\]

8. Substituting \( \lambda = 1, f(x) = 3 - 2x, \) and \( K(x, t) = 1 \) into the formula yields

\[
u(x) = 3 - 2x + \int_0^x (3 - 2t) dt + \int_0^x \int_0^t (3 - 2t)(3 - 2t_1) dt_1 dt + \cdots
\]

Accordingly we obtain \( u(x) \) in a series form

\[
u(x) = 3 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots
\]

and in a closed form, \( u(x) \) is given by

\[
u(x) = 2 + e^x
\]

9. Substituting \( \lambda = -1, f(x) = 2 + \frac{1}{2!}x^2, \) and \( K(x, t) = (x - t) \) into the formula yields

\[
u(x) = 2 + \frac{1}{2!}x^2 - \int_0^x (x - t) (2 + \frac{1}{2!}t^2) dt
\]

\[
+ \int_0^x \int_0^t (x - t)(t - t_1)(2 + \frac{1}{2!}t_1^2) dt_1 dt + \cdots
\]

Accordingly we obtain \( u(x) \) in a series form

\[
u(x) = 2 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots
\]

and in a closed form, \( u(x) \) is given by

\[
u(x) = 1 + \cos x
\]

10. Substituting \( \lambda = -1, f(x) = 1 - x + \frac{1}{2!}x^2, \) and \( K(x, t) = (x - t) \) into the formula yields
Chapter 3. Volterra Integral Equations

\[ u(x) = 1 - x + \frac{1}{2!}x^2 - \int_0^x (x-t) \left(1 + t + \frac{1}{2!}t^2\right) dt \\
+ \int_0^x \int_0^t (x-t)(t-t_1) \left(1 + t_1 + \frac{1}{2!}t_1^2\right) dt_1 dt + \cdots \]

Accordingly we obtain \( u(x) \) in a series form

\[ u(x) = 1 - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \right) \]
and in a closed form, \( u(x) \) is given by

\[ u(x) = 1 - \sin x \]

11. Substituting \( \lambda = 1, f(x) = 2 - \frac{1}{2!}x^2, \) and \( K(x, t) = (x-t) \) into the formula yields

\[ u(x) = 2 - \frac{1}{2!}x^2 - \int_0^x (x-t) \left(2 - \frac{1}{2!}t^2\right) dt \\
+ \int_0^x \int_0^t (x-t)(t-t_1) \left(2 - \frac{1}{2!}t_1^2\right) dt_1 dt + \cdots \]

Accordingly we obtain \( u(x) \) in a series form

\[ u(x) = 1 + \left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \right) \]
and in a closed form, \( u(x) \) is given by

\[ u(x) = 1 + \cosh x \]

12. Substituting \( \lambda = 1, f(x) = \frac{1}{2!}x^2, \) and \( K(x, t) = (x-t) \) into the formula yields

\[ u(x) = \frac{1}{2!}x^2 + \int_0^x (x-t) \left(\frac{1}{2!}t^2\right) dt + \int_0^x \int_0^t (x-t)(t-t_1) \left(\frac{1}{2!}t_1^2\right) dt_1 dt + \cdots \]

Accordingly we obtain \( u(x) \) in a series form

\[ u(x) = \left(\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \right) \]
and in a closed form, \( u(x) \) is given by

\[ u(x) = -1 + \cosh x \]
3.9 Volterra Equations of the First Kind

Exercises 3.9

1. Noting that \( K(x, t) = 5 + 3x - 3t \), hence \( K(x, x) = 5 \neq 0 \)

Differentiating both sides of the equation with respect to \( x \) yields

\[
10x + 3x^2 = 5u(x) + \int_0^x 3u(t)dt
\]

or equivalently

\[
u(x) = 2x + \frac{3}{5}x^2 - \frac{3}{5} \int_0^x u(t)dt
\]

Using the modified decomposition method, we set

\[
u_0(x) = 2x
\]

which gives

\[
u_1(x) = \frac{3}{5}x^2 - \frac{3}{5} \int_0^x 2t dt
\]

Consequently, we find

\[
u_k(x) = 0 \quad \text{for} \quad k \geq 1
\]

Accordingly, the exact solution is

\[
u(x) = 2x
\]

2. Noting that \( K(x, t) = e^{t-x} \), hence \( K(x, x) = 1 \neq 0 \)

Differentiating both sides of the equation with respect to \( x \) yields

\[
e^{-x} - xe^{-x} = u(x) - \int_0^x e^{t-x}u(t)dt
\]

or equivalently

\[
u(x) = e^{-x} - xe^{-x} + \int_0^x e^{t-x}u(t)dt
\]
Chapter 3. Volterra Integral Equations

Using the modified decomposition method, we set

\[ u_0(x) = e^{-x} \]

which gives

\[ u_1(x) = -xe^{-x} + \int_0^x e^{-x} \, dt \]

Consequently, we find

\[ u_k(x) = 0 \quad \text{for} \ k \geq 1 \]

Accordingly, the exact solution is

\[ u(x) = e^{-x} \]

3. Noting that \( K(x, t) = 1 + x - t \), hence \( K(x, x) = 1 \neq 0 \)

Differentiating both sides of the equation with respect to \( x \) yields

\[ 2e^x - 1 = u(x) + \int_0^x u(t) \, dt \]

or equivalently

\[ u(x) = 2e^x - 1 - \int_0^x u(t) \, dt \]

Using the modified decomposition method, we set

\[ u_0(x) = e^x \]

which gives

\[ u_1(x) = e^x - 1 - \int_0^x e^t \, dt \]

Consequently, we find

\[ u_k(x) = 0 \quad \text{for} \ k \geq 1 \]
3.9. *Volterra Equations of the First Kind*

Accordingly, the exact solution is

\[ u(x) = e^x \]

4. Noting that \( K(x, t) = 2 - x + t \), hence \( K(x, x) = 2 \neq 0 \)

Differentiating both sides of the equation with respect to \( x \) yields

\[ 2 \sinh x - \cosh x + 1 = 2u(x) - \int_0^x u(t)dt \]

or equivalently

\[ u(x) = \sinh x - \frac{1}{2} \cosh x + \frac{1}{2} + \frac{1}{2} \int_0^x u(t)dt \]

Using the *modified decomposition method*, we set

\[ u_0(x) = \sinh x \]

which gives

\[ u_1(x) = \frac{1}{2} \cosh x + \frac{1}{2} + \frac{1}{2} \int_0^x \sinh tdt \]

Consequently, we find

\[ u_k(x) = 0 \quad \text{for} \quad k \geq 1 \]

Accordingly, the exact solution is

\[ u(x) = \sinh x \]

5. Noting that \( K(x, t) = 4 + 3x - 3t \), hence \( K(x, x) = 4 \neq 0 \)

Differentiating both sides of the equation with respect to \( x \) yields

\[ 4 \cos x + 3 \sin x = 4u(x) + 3 \int_0^x u(t)dt \]

or equivalently

\[ u(x) = \cos x + \frac{3}{4} \sin x - \frac{3}{4} \int_0^x u(t)dt \]
Using the modified decomposition method, we set

\[ u_0(x) = \cos x \]

which gives

\[ u_1(x) = \frac{3}{4} \sin x - \frac{3}{4} \int_0^x \cos t \, dt \]

Consequently, we find

\[ u_k(x) = 0 \quad \text{for} \quad k \geq 1 \]

Accordingly, the exact solution is

\[ u(x) = \cos x \]

6. Noting that \( K(x, t) = 1 + x - t \), hence \( K(x, x) = 1 \neq 0, \quad x < \frac{\pi}{2} \)

Differentiating both sides of the equation with respect to \( x \) yields

\[ \sec^2 x + \tan x = u(x) + \int_0^x u(t) \, dt \]

or equivalently

\[ u(x) = \sec^2 x + \tan x - \int_0^x u(t) \, dt \]

Using the modified decomposition method, we set

\[ u_0(x) = \sec^2 x \]

which gives

\[ u_1(x) = \tan x - \int_0^x \sec^2 t \, dt \]

Consequently, we find

\[ u_k(x) = 0 \quad \text{for} \quad k \geq 1 \]
3.9. Volterra Equations of the First Kind

Accordingly, the exact solution is

\[ u(x) = \sec^2 x \]
Chapter 4

Fredholm
Integro-Differential
Equations

4.3 The Direct Computation Method

Exercises 4.3

1. \( u'(x) = \frac{1}{6} + \frac{5}{36}x - \int_0^1 xt u(t) dt, \ u(0) = \frac{1}{6} \)

We first set

\[ \alpha = \int_0^1 tu(t) dt \]

so that the given equation can be rewritten as

\[ u'(x) = \frac{1}{6} + \left( \frac{5}{36} - \alpha \right)x \]

Integrating both sides from 0 to \( x \) and using the given condition gives

\[ u(x) = \frac{1}{6}(1 + x) + \left( \frac{5}{36} - \alpha \right)\frac{x^2}{2} \]

Substituting this expression of \( u(x) \) into the first equation for \( \alpha \) yields
Chapter 4. Fredholm Integro-Differential Equations

\[ \alpha = \int_0^1 t \left( \frac{1}{6} (1 + t) + \left( \frac{5}{36} - \frac{\alpha}{2} \right) t^2 \right) dt \]

so that

\[ \alpha = \frac{5}{36} \]

This gives

\[ u(x) = \frac{1}{6} (1 + x) \]

2. \[ u'(x) = \frac{1}{21} x - \int_0^1 xt u(t) dt, \quad u(0) = \frac{1}{6} \]

We first set

\[ \alpha = \int_0^1 t u(t) dt \]

so that the given equation can be rewritten as

\[ u'(x) = \frac{1}{21} x - \alpha x \]

Integrating both sides from 0 to \( x \) and using the given condition give

\[ u(x) = \frac{1}{6} + \left( \frac{1}{42} - \frac{\alpha}{2} \right) x^2 \]

Substituting this expression of \( u(x) \) into the first equation for \( \alpha \) yields

\[ \alpha = \int_0^1 t \left( \frac{1}{6} + \left( \frac{1}{42} - \frac{\alpha}{2} \right) t^2 \right) dt \]

so that

\[ \alpha = \frac{5}{63} \]

This gives

\[ u(x) = \frac{1}{6} - \frac{1}{63} x^2 \]

3. \[ u''(x) = -\sin x + x - \int_0^{\pi/2} xt u(t) dt, \quad u(0) = 0, \quad u'(0) = 1 \]
4.3. The Direct Computation Method

We first set
\[ \alpha = \int_0^{\pi/2} tu(t)dt \]
so that the given equation can be rewritten as
\[ u''(x) = -\sin x + (1 - \alpha)x \]

Integrating both sides from 0 to \( x \) twice and using the given conditions gives
\[ u'(x) = \cos x + \frac{1 - \alpha}{2!}x^2 \]
and
\[ u(x) = \sin x + \frac{1 - \alpha}{3!}x^3 \]

Substituting this expression of \( u(x) \) into the first equation for \( \alpha \) yields
\[ \alpha = \int_0^{\pi/2} t \left( \sin t + \frac{1 - \alpha}{3!}t^3 \right) dt \]
so that
\[ \alpha = 1 \]

This gives
\[ u(x) = \sin x \]

4. \[ u''(x) = \frac{2}{3} - \frac{1}{3}x + \int_0^{\frac{1}{x}}(x - t)u(t)dt, \quad u(0) = u'(0) = 0 \]

We first set
\[ \alpha = \int_0^{\frac{1}{x}} u(t)dt \]
and
\[ \beta = \int_0^{\frac{1}{x}} tu(t)dt \]
so that the given equation can be rewritten as

$$u''(x) = \frac{9}{4} - \frac{x}{3} + \alpha x - \beta$$

Integrating both sides from 0 to $x$ twice and using the given conditions gives

$$u'(x) = \frac{9}{4} x - \frac{1}{6} x^2 + \frac{\alpha}{2} x^2 - \beta x$$

and

$$u(x) = \left(\frac{9}{8} - \frac{\beta}{2}\right) x^2 + \left(\frac{\alpha}{6} - \frac{1}{18}\right) x^3$$

Substituting this expression of $u(x)$ into the first equation for $\alpha$ yields

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1}{4}$$

This gives

$$u(x) = x^2$$

5. $u'(x) = 2\sec^2 x \tan x - x + \int_0^{\pi/4} xu(t)dt, \ u(0) = 1$

We first set

$$\alpha = \int_0^{\pi/4} u(t)dt$$

so that the given equation can be rewritten as

$$u'(x) = 2\sec^2 x \tan x + (\alpha - 1) x$$

Integrating both sides from 0 to $x$ and using the given condition gives

$$u(x) = \sec^2 x + \frac{\alpha - 1}{2} x^2$$

Substituting this expression of $u(x)$ into the first equation for $\alpha$ yields

$$\alpha = 1$$
4.3. The Direct Computation Method

This gives

\[ u(x) = \sec^2 x \]

6. Integrating both sides from 0 to \( x \) and using the given condition gives

\[ u(x) = 1 - (5 + \beta)x + \left( \frac{\alpha}{2} - 3 \right)x^2 \]

where

\[ \alpha = \int_{-1}^{1} u(t) \, dt, \quad \beta = \int_{-1}^{1} tu(t) \, dt \]

Substituting \( u(x) \) into \( \alpha \) and \( \beta \) and by solving the resulting equations we find \( \alpha = 0, \beta = -2 \) hence we find

\[ u(x) = 1 - 3x - 3x^2 \]

7. Integrating both sides from 0 to \( x \) and using the given condition gives

\[ u(x) = \sin x + \left( \frac{\alpha}{2} - \frac{1}{2} \right)x^2 + (1 - \beta)x \]

where

\[ \alpha = \int_{0}^{\pi/2} u(t) \, dt, \quad \beta = \int_{0}^{\pi/2} tu(t) \, dt \]

Proceeding as before we find \( \alpha = 1, \beta = 1 \) hence we find

\[ u(x) = \sin x \]

8. Integrating both sides from 0 to \( x \) and using the given condition gives

\[ u(x) = \sin x + \cos x \left( \frac{\alpha}{2} - x^2 + \left( \frac{\pi}{2} - \beta \right)x \right) \]

where

\[ \alpha = \int_{0}^{\pi/2} u(t) \, dt, \quad \beta = \int_{0}^{\pi/2} tu(t) \, dt \]

Proceeding as before we find \( \alpha = 2, \beta = \frac{\pi}{2} \) hence we find

\[ u(x) = \sin x + \cos x \]
4.4 The Adomian Decomposition Method

Exercises 4.4

1. \( u'(x) = \sinh x + \frac{1}{8}(1 - e^{-1})x - \frac{1}{8}x \int_0^1 tu(t)dt, \ u(0) = 1 \)

Integrating both sides from 0 to \( x \) and using the given condition gives

\[ u(x) = \cosh x + \frac{1}{16}(1 - e^{-1})x^2 - \frac{1}{16}x^2 \int_0^1 tu(t)dt \]

Following the Adomian decomposition method, we set

\[ u_0(x) = \cosh x + \frac{1}{16}(1 - e^{-1})x^2 \]

\[ u_1(x) = -\frac{1}{16}x^2 \int_0^1 t \left( \cosh t + \frac{1}{16}(1 - e^{-1})t^2 \right) dt \]

so that

\[ u_1(x) = -\frac{1}{16}(1 - e^{-1})x^2 - \frac{1}{16 \times 64}(1 - e^{-1})x^2 \]

Cancelling the noise term of \( u_0 \) that appears in \( u_1 \), and justifying that the remaining non-canceled term in \( u_0 \) justifies the equation, we therefore find

\[ u(x) = \cosh x \]

2. \( u'(x) = 1 - \frac{1}{3}x + x \int_0^1 tu(t)dt, \ u(0) = 0 \)

Integrating both sides from 0 to \( x \) and using the given condition gives

\[ u(x) = x - \frac{1}{6}x^2 + \frac{1}{2}x^2 \int_0^1 tu(t)dt \]

Following the Adomian decomposition method, we set

\[ u_0(x) = x - \frac{1}{6}x^2 \]

\[ u_1(x) = \frac{1}{2}x^2 \int_0^1 t \left( t - \frac{1}{6}t^2 \right) dt \]
4.4. The Adomian Decomposition Method

so that

\[ u_1(x) = \frac{7}{48} x^2 \]

and continuing in the same manner we find

\[ u_2(x) = \frac{7}{48 \times 8} x^2 \]

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

\[ u(x) = x - \frac{1}{6} x^2 + \frac{7}{48} x^2 \left( 1 + \frac{1}{8} + \cdots \right) \]

we therefore find

\[ u(x) = x \]

3. \[ u'(x) = xe^x + e^x - x + x \int_0^1 u(t) \, dt, \quad u(0) = 0 \]

Integrating both sides from 0 to \( x \) and using the given condition gives

\[ u(x) = xe^x - \frac{1}{2} x^2 + \frac{1}{2} x^2 \int_0^1 u(t) \, dt \]

Following the Adomian decomposition method, we set

\[ u_0(x) = xe^x - \frac{1}{2} x^2 \]

\[ u_1(x) = \frac{5}{12} x^2 \]

and continuing in the same manner we find

\[ u_2(x) = \frac{5}{72} x^2 \]

\[ u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \]

\[ u(x) = xe^x - \frac{1}{2} x^2 + \frac{5}{72} x^2 \left( 1 + \frac{1}{6} + \frac{1}{36} + \cdots \right) \]

we therefore find

\[ u(x) = xe^x \]

4. \[ u'(x) = x \cos x + \sin x - x + x \int_0^\frac{x}{2} u(t) \, dt, \quad u(0) = 0 \]
Chapter 4. Fredholm Integro-Differential Equations

Integrating both sides from 0 to \( x \) and using the given condition give

\[ u(x) = x \sin x - \frac{1}{2} x^2 + \frac{1}{2} x^2 \int_0^x u(t) dt \]

Following the Adomian decomposition method, we set

\[ u_0(x) = x \sin x - \frac{1}{2} x^2 \]
\[ u_1(x) = \frac{1}{2} x^2 \]

Cancelling the noise terms between \( u_0(x) \) and \( u_1(x) \) and justifying that the non-canceled term of \( u_0(x) \) satisfies the integral equation gives

\[ u(x) = x \sin x \]

5. \( u''(x) = -\sin x + x - x \int_0^x tu(t) dt \), \( u(0) = 0, u'(0) = 1 \)

Integrating both sides from 0 to \( x \) twice and using the given conditions give

\[ u(x) = \sin x + \frac{1}{3!} x^3 - \frac{1}{3!} x^3 \int_0^x tu(t) dt \]

Following the Adomian decomposition method, we set

\[ u_0(x) = \sin x + \frac{1}{3!} x^3 \]
\[ u_1(x) = -\frac{1}{3!} x^3 - \frac{\pi^5}{160 \times (3!)^2} x^3 \]

Cancelling the noise terms between \( u_0(x) \) and \( u_1(x) \) and justifying that the non-canceled term of \( u_0(x) \) satisfies the integral equation gives

\[ u(x) = \sin x \]

6. \( u'''(x) = 6 + x - x \int_0^1 u''(t) dt \) \( u(0) = -1, u'(0) = 1, u''(0) = -2 \)

Integrating both sides from 0 to \( x \) three times and using the given conditions give

\[ u(x) = -1 + x - x^2 + x^3 + \frac{1}{4} x^3 - \frac{1}{4!} x^4 \int_0^1 u''(t) dt \]

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4.4. The Adomian Decomposition Method

Following the Adomian decomposition method, we set

\[ u_0(x) = -1 + x - x^2 + x^3 + \frac{1}{4!} x^4 \]
\[ u_1(x) = -\frac{1}{4!} x^4 - \frac{1}{4! \times 3!} x^4 \]

Cancelling the noise terms between \( u_0(x) \) and \( u_1(x) \) and justifying that the non-cancelled term of \( u_0(x) \) satisfies the integral equation gives

\[ u(x) = x^3 - x^2 + x - 1 \]

7. \( u''(x) = -\cos x + x + \int_0^x x u''(t) dt \), \( u(0) = u''(0) = 0, u'(0) = 1 \)

Integrating both sides from 0 to \( x \) three times and using the given conditions give

\[ u(x) = \sin x + \frac{1}{3!} x^4 + \frac{1}{4!} x^4 \int_0^x u''(t) dt \]

Following the Adomian decomposition method, we set

\[ u_0(x) = \sin x + \frac{1}{4!} x^4 \]
\[ u_1(x) = -\frac{1}{4!} x^4 + \frac{\pi^3}{4! \times 48} x^4 \]

Cancelling the noise terms between \( u_0(x) \) and \( u_1(x) \) and justifying that the non-cancelled term of \( u_0(x) \) satisfies the integral equation gives

\[ u(x) = \sin x \]

8. Integrating both sides from 0 to \( x \) and using the given condition we find

\[ u(x) = \sin x + \cos x - x^2 + \frac{x}{2} + \int_0^x \left( \frac{x^2}{2} - xt \right) u(t) dt \]

Using the modified decomposition method, we set

\[ u_0(x) = \sin x + \cos x \]
\[ u_1(x) = 0 \]
\[ u(x) = \sin x + \cos x \]
Chapter 4. Fredholm Integro-Differential Equations

9. Integrating both sides from 0 to \( x \) and using the given condition we find

\[
u(x) = \cos x - \frac{x^2}{2} + \frac{x}{2} - x + \int_0^x \left( \frac{x^2}{2} - xt \right) u(t) dt
\]

Using the modified decomposition method, we set

\[
u_0(x) = \cos x
\]
\[
u_1(x) = 0
\]
\[
u(x) = \cos x
\]

10. Integrating both sides from 0 to \( x \) and using the given condition we find

\[
u(x) = \sin x - \cos x + 2x \frac{x}{2} + \int_0^\pi \left( \frac{x^2}{2} - xt \right) u(t) dt
\]

Using the modified decomposition method, we set

\[
u_0(x) = \sin x - \cos x
\]
\[
u_1(x) = 0
\]
\[
u(x) = \sin x - \cos x
\]

4.5 The Variational Iteration Method

Exercises 4.5

1. The correction functional for this equation is given by

\[
u_{n+1}(x) = \nu_n(x) - \int_0^x \left( \nu_n(t) - \sin t - t \cos t + 1 - \int_0^{\pi/2} \nu_n(r) dr \right) dt
\]

Using \( \nu_0(x) = 0 \) into the correction functional gives the following successive approximations

\[
u_0(x) = 0
\]
\[
u_1(x) = \nu_0(x) - \int_0^x \left( \nu_0(t) - \sin t - t \cos t + 1 - \int_0^{\pi/2} \nu_0(r) dr \right) dt = x \sin x - x
\]
\[
u_2(x) = \nu_1(x) - \int_0^x \left( \nu_1(t) - \sin t - t \cos t + 1 - \int_0^{\pi/2} \nu_1(r) dr \right) dt = (x \sin x - x) + (x - \frac{\pi^2}{8} x)
\]
4.5. The Variational Iteration Method

\[ u_3(x) = (x \sin x - x) + (x - \frac{\pi^2}{8} x) + \left(\frac{\pi^2}{8} x - \cdots\right) \]
\[ u(x) = x \sin x \]

2. The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) - \cos t + t \sin t - 2 - \int_0^t u_n(r) \, dr \right) \, dt \]

Using \( u_0(x) = 0 \) into the correction functional gives the following successive approximations

\[ u_0(x) = 0 \]
\[ u_1(x) = x \cos x + 2x \]
\[ u_2(x) = (x \cos x + 2x) + (\pi^2 x - 2x) \]
\[ u_3(x) = (x \cos x + 2x) + (\pi^2 x - 2x) + (\frac{\pi^4}{8} x - \pi^2 x) + \cdots \]
\[ u(x) = x \cos x \]

3. The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) - 2 \sec^2 x \tan x + \frac{\pi^2}{8} - \int_0^t u_n(r) \, dr \right) \, dt \]

Using \( u_0(x) = 1 \) into the correction functional gives the following successive approximations

\[ u_0(x) = 1 \]
\[ u_1(x) = \sec^2 x + \frac{\pi}{4} x - \frac{\pi^2}{16} x \]
\[ u_2(x) = \sec^2 x + (x + \frac{\pi}{4} x) - (\frac{\pi}{4} x + \frac{\pi^2}{32} x) \]
\[ u_3(x) = \sec^2 x + x + \cdots \]
\[ u(x) = x + \sec^2 x \]

4. The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) - 3 + 12 t - \int_0^t r u_n(r) \, dr \right) \, dt \]

Using \( u_0(x) = 1 \) into the correction functional gives the following successive approximations

\[ u_0(x) = 1 \]
\[ u_1(x) = 1 - 6x^2 + \frac{7}{2} x \]
\[ u_2(x) = 1 - 6x^2 + \frac{19}{6} x \]
\[ u_3(x) = 1 - 6x^2 + \frac{55}{18} x \]
\[ u(x) = 1 + 3x - 6x^2 \]

5. The correction functional for this equation is given by
Chapter 4. Fredholm Integro-Differential Equations

$$u_{n+1}(x) = u_n(x) - \int_0^x \left( u_n'(t) - e^t + 1 - \int_0^1 ru_n(r) \, dr \right) \, dt$$

Using $u_0(x) = 1$ into the correction functional gives the following successive approximations

- $u_0(x) = 1$
- $u_1(x) = e^x - \frac{1}{2} x$
- $u_2(x) = e^x - \frac{1}{6} x$
- $u_3(x) = e^x - \frac{1}{18} x$
- $u_n(x) = e^x - \frac{1}{2x^{3n-1}} x$
- $u(x) = e^x$

6. The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left( u_n'(t) - 10 \frac{t}{t^2} - 2t + 3t^2 - \int_0^1 ru_n(r) \, dr \right) \, dt$$

Using $u_0(x) = 1$ into the correction functional gives the following successive approximations

- $u_0(x) = 1$
- $u_1(x) = 1 + x^2 - x^3 - \frac{7}{12} x$
- $u_2(x) = 1 + x^2 - x^3 - \frac{19}{21} x$
- $u_3(x) = 1 + x^2 - x^3 - \frac{43}{42} x$
- $u(x) = 1 - x + x^2 - x^3$

4.6 Converting to Fredholm Integral Equations

Exercises 4.6

1. $u'(x) = -x \sin x + \cos x + (1 - \frac{x}{2})x + \int_0^x xu(t) \, dt$, $u(0) = 0$

Integrating both sides from 0 to $x$ and using the given condition give

$$u(x) = x \cos x + (1 - \frac{x}{2}) \frac{x^2}{2} + \frac{x^2}{2} \int_0^x u(t) \, dt$$

or equivalently

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4.6. Converting to Fredholm Integral Equations

\[ u(x) = x \cos x + (1 - \frac{\pi}{2}) \frac{x^2}{2} + \alpha \frac{x^2}{2} \]

where

\[ \alpha = \int_{0}^{\frac{\pi}{2}} u(t) dt \]

Substituting for \( u(x) \) from the above equation yields

\[ \alpha = -\left(1 - \frac{\pi}{2}\right) \]

so that

\[ u(x) = x \cos x \]

2. \( u''(x) = -e^x + \frac{1}{2} x + \int_{0}^{1} xt u(t) dt, \quad u(0) = 0, \quad u'(0) = -1 \)

Integrating both sides from 0 to \( x \) twice and using the given conditions give

\[ u(x) = -e^x + 1 + \frac{x^3}{12} + \frac{x^3}{6} \int_{0}^{1} tu(t) dt \]

or equivalently

\[ u(x) = -e^x + 1 + \frac{x^3}{12} + \alpha \frac{x^3}{6} \]

where

\[ \alpha = \int_{0}^{1} tu(t) dt \]

Substituting for \( u(x) \) from the above equation yields

\[ \alpha = -\frac{1}{2} \]

so that

\[ u(x) = 1 - e^x \]

3. \( u''(x) = -\sin x + \cos x + \frac{\pi}{2} x - \int_{0}^{\frac{\pi}{2}} xt u(t) dt, \quad u(0) = -1, \quad u'(0) = 1 \)
Chapter 4. Fredholm Integro-Differential Equations

Integrating both sides from 0 to $x$ twice and using the given conditions give

$$ u(x) = \sin x - \cos x + \left(2 - \frac{\pi}{2}\right)\frac{x^3}{6} - \frac{x^3}{6} \int_0^x t u(t) \, dt, $$

or equivalently

$$ u(x) = \sin x - \cos x + \left(2 - \frac{\pi}{2}\right)\frac{x^3}{6} - \frac{x^3}{6} \alpha, $$

where

$$ \alpha = \int_0^x t u(t) \, dt, $$

Substituting for $u(x)$ from the above equation yields

$$ \alpha = (2 - \frac{\pi}{2}) $$

so that

$$ u(x) = \sin x - \cos x $$

4. $u'(x) = \frac{7}{6} - 11x + \int_0^1 (x - t)u(t) \, dt, \quad u(0) = 0$

Integrating both sides from 0 to $x$ and using the given condition give

$$ u(x) = \frac{7}{6}x - \frac{11}{2}x^2 + \frac{1}{2}x^2 \int_0^1 u(t) \, dt - x \int_0^1 t u(t) \, dt, $$

or equivalently

$$ u(x) = \frac{7}{6}x - \frac{11}{2}x^2 + \alpha \frac{1}{2}x^2 - x \beta, $$

where

$$ \alpha = \int_0^1 u(t) \, dt, $$

and

$$ \beta = \int_0^1 t u(t) \, dt, $$
4.6. Converting to Fredholm Integral Equations

Substituting for $u(x)$ from the above equation yields

\[ \alpha = -1 \]
\[ \beta = -\frac{5}{6} \]

so that

\[ u(x) = 2x - 6x^2 \]

5. $u'(x) = \frac{1}{4}x + \cos 2x - \int_0^{\frac{\pi}{4}} xu(t)dt$, $u(0) = 0$

Integrating both sides from 0 to $x$ and using the given condition give

\[ u(x) = \frac{1}{8}x^2 + \frac{1}{2}\sin 2x - \frac{1}{2}x^2 \int_0^{\frac{\pi}{4}} u(t)dt, \]

or equivalently

\[ u(x) = \frac{1}{8}x^2 + \frac{1}{2}\sin 2x - \frac{1}{2}\alpha x^2 \]

where

\[ \alpha = \int_0^{\frac{\pi}{4}} u(t)dt \]

Substituting for $u(x)$ from the above equation yields

\[ \alpha = \frac{1}{4} \]

so that

\[ u(x) = \frac{1}{2}\sin x \]
Chapter 5

Volterra
Integro-Differential
Equations

5.3 The Series Solution Method

Exercises 5.3

1. \( u'(x) = 1 - 2x \sin x + \int_0^x u(t) \, dt, \ u(0) = 0 \)

Using the series method, noting that \( a_0 = 0 \) by using the given condition, we find

\( u(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \)

and hence

\( u'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \)

Substituting into both sides of the equation, integrating the right hand side and equating the coefficients of like terms of \( x \) in both sides give

\( a_0 = 0, \ a_1 = 1, \ a_2 = 0, \ a_3 = -\frac{1}{2!}, \ a_4 = 0, \ a_5 = \frac{1}{4!} \)

Accordingly, \( u(x) \) in a series form is given by

\( u(x) = x(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots) \)

and in a closed form

\( u(x) = x \cos x \)

2. \( u'(x) = -1 + \frac{1}{2}x^2 - xe^x - \int_0^x tu(t) \, dt, \ u(0) = 0 \)

Using the series method, noting that \( a_0 = 0 \) by using the given condition,
3. \( u''(x) = 1 - x(\cos x + \sin x) - \int_0^x tu(t) \, dt, \ u(0) = -1, \ u'(0) = 1 \)

Using the series method, noting that \( a_0 = -1, a_1 = 1 \) by using the given conditions, we find

\[
u(x) = -1 + x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots
\]

and hence

\[
u'(x) = 1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots
\]

\[
u''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots
\]

Substituting into both sides of the equation, using the Taylor expansions of \( \sin x \) and \( \cos x \), integrating the right hand side and equating the coefficients of like terms of \( x \) in both sides give

\[
a_0 = -1, \ a_1 = 1, \ a_2 = -\frac{1}{3}, \ a_3 = -\frac{1}{6}, \ a_4 = -\frac{1}{10}
\]

Accordingly, \( u(x) \) in a series form is given by

\[
u(x) = -(x + \frac{1}{3}x^2 + \frac{1}{6}x^3 + \frac{1}{10}x^4 + \cdots)
\]

and in a closed form

\[
u(x) = 1 - e^x
\]
5. \( u''(x) = \frac{1}{2} x^2 - x \cosh x - \int_0^x t u(t) dt, \) \( u(0) = 1, \) \( u'(0) = -1 \)

Using the series method, noting that \( a_0 = 1, a_1 = -1 \) by using the given conditions, we find

\[
u(x) = 1 - x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots
\]

and hence

\[
u'(x) = -1 + 2 a_2 x + 3 a_3 x^2 + 4 a_4 x^3 + \cdots
\]
\[
u''(x) = 2 a_2 + 6 a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots
\]

Substituting into both sides of the equation, using the Taylor expansions of \( \cosh x \), integrating the right hand side and equating the coefficients of like terms of \( x \) in both sides give

\[
a_0 = 1, \quad a_1 = -1, \quad a_2 = 0, \quad a_3 = -\frac{1}{3!}, \quad a_4 = 0, \quad a_5 = -\frac{1}{5!}
\]

Accordingly, \( u(x) \) in a series form is given by

\[
u(x) = 1 - (x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots)
\]

and in a closed form

\[
u(x) = 1 - \sinh x
\]

5.4 The Adomian Decomposition Method

Exercises 5.4

1. \( u''(x) = 1 + x - \frac{1}{3!} x^3 + \int_0^x (x - t) u(t) dt, \) \( u(0) = 1, \) \( u'(0) = 2 \)

Applying \( L^{-1} \) to both sides and using the initial conditions we find

\[
u(x) = 1 + 2 x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 - \frac{1}{5!} x^5 + L^{-1} \left( \int_0^x (x - t) u(t) dt \right)
\]

where

\[
L^{-1}(\cdot) = \int_0^x \int_0^t (\cdot) dt \, dt
\]

Using the Adomian decomposition method where we set

\[
u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots
\]

Hence we obtain

\[
u_0(x) = 1 + 2 x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 - \frac{1}{5!} x^5
\]
\[
u_1(x) = L^{-1} \left( \int_0^x (x - t) \left( 1 + 2 t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 - \frac{1}{5!} t^5 \right) dt \right)
\]

so that

\[
u_1(x) = \frac{1}{3!} x^4 + \frac{1}{5!} x^5 + \frac{1}{7!} x^6 + \cdots
\]

Accordingly \( u(x) \) in a series form is given by

\[
u(x) = x + \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \right)
\]

and in a closed form

\[
u(x) = x + e^x
\]
Chapter 5. Volterra Integro-Differential Equations

2. \( u''(x) = -1 - \frac{1}{2}x^2 + \int_0^x (x-t)u(t)dt, \quad u(0) = 2, \quad u'(0) = 0 \)

Applying \( L^{-1} \) to both sides and using the initial conditions we find
\( u(x) = 2 - \frac{1}{2}x^2 - \frac{1}{4}x^4 + L^{-1}\left(\int_0^x (x-t)u(t)dt\right) \)

where
\( L^{-1}(.) = \int_0^x \int_0^x (-) \, dt \)

Using the Adomian decomposition method where we set
\( u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \)

Hence we obtain
\( u_0(x) = 2 - \frac{1}{2}x^2 - \frac{1}{4}x^4 \)
\( u_1(x) = L^{-1}\left(\int_0^x (x-t)(2 - \frac{1}{2}t^2 - \frac{1}{4}t^4)dt\right) \)

so that
\( u_1(x) = \frac{1}{12}x^4 - \frac{1}{5!}x^6 - \frac{1}{8!}x^8 \)

Accordingly \( u(x) \) in a series form is given by
\( u(x) = 1 + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) \)

and in a closed form
\( u(x) = 1 + \cos x \)

3. \( u'(x) = 2 + \int_0^x u(t)dt, \quad u(0) = 2 \)

Applying \( L^{-1} \) to both sides and using the initial conditions we find
\( u(x) = 2 + 2x + L^{-1}\left(\int_0^x u(t)dt\right) \)

where
\( L^{-1}(.) = \int_0^x \int_0^x (-) \, dt \)

Using the Adomian decomposition method where we set
\( u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \)

Hence we obtain
\( u_0(x) = 2 + 2x \)
\( u_1(x) = x^2 + \frac{1}{4}x^3 \)

Accordingly \( u(x) \) in a series form is given by
\( u(x) = 2 \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \)

and in a closed form
\( u(x) = 2e^x \)

4. \( u'(x) = 1 - \int_0^x u(t)dt, \quad u(0) = 1 \)

Applying \( L^{-1} \) to both sides and using the initial conditions we find
\( u(x) = 1 + x - L^{-1}\left(\int_0^x u(t)dt\right) \)

where
5.5. The Variational Iteration Method

5.5 The Variational Iteration Method

Exercises 5.5

1. The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u_n(t) + t - \frac{1}{2} t^2 - \int_0^t u_n(r) \, dr \right) \, dt. \]

We find the following successive approximations:

\[ u_0(x) = 0, \]
Chapter 5. Volterra Integro-Differential Equations

\[ u_1(x) = u_0(x) - \int_0^x \left( u_0'(t) - 2e^t + 1 - \int_0^t u_0(r) \, dr \right) \, dt \]
\[ = -2 - x + 2e^x, \]
\[ u_2(x) = u_1(x) - \int_0^x \left( u_1'(t) - 2e^t + 1 - \int_0^t u_1(r) \, dr \right) \, dt, \]
\[ = x + x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{1}{4}x^5 + \cdots, \]
\[ u_3(x) = u_2(x) - \int_0^x \left( u_2'(t) - 2e^t + 1 - \int_0^t u_2(r) \, dr \right) \, dt \]
\[ = x + x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{1}{5}x^5 + \frac{1}{7}x^6 + \cdots. \]

This gives the exact solution
\[ u(x) = xe^x. \]

2. The correction functional for this equation is given by
\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u_n'(t) - 2 \cos t + \frac{1}{2}t^2 - \int_0^t u_n(r) \, dr \right) \, dt. \]
We find the following successive approximations:
\[ u_0(x) = 0, \]
\[ u_1(x) = u_0(x) - \int_0^x \left( u_0'(t) - 2 \cos t + \frac{1}{2}t^2 - \int_0^t u_0(r) \, dr \right) \, dt \]
\[ = 2 \sin x - \frac{1}{3!}x^3, \]
\[ u_2(x) = u_1(x) - \int_0^x \left( u_1'(t) - 2 \cos t + \frac{1}{2}t^2 - \int_0^t u_1(r) \, dr \right) \, dt \]
\[ = 2x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots. \]
This gives the exact solution
\[ u(x) = x + \sin x. \]

3. Proceeding as before we find the following successive approximations:
\[ u_0(x) = -1, \]
\[ u_1(x) = -1 + x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 - \frac{1}{4}x^4, \]
\[ u_2(x) = -1 + x + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots. \]
This gives the exact solution
\[ u(x) = x - \cos x. \]

4. Proceeding as before we find the following successive approximations:
\[ u_0(x) = 1, \]
\[ u_1(x) = 1 + x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3, \]
\[ u_2(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^5 + \cdots. \]
This gives the exact solution
\[ u(x) = x + \cosh x. \]

5. Proceeding as before we find the following successive approximations:
5.6. Converting to Volterra Equations

\( u_0(x) = 1 + x, \)
\( u_1(x) = 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \)
\( u_2(x) = 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots. \)

This gives the exact solution
\( u(x) = x + e^x. \)

6. Proceeding as before we find the following successive approximations:
\( u_0(x) = 1 - x, \)
\( u_1(x) = 1 - x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \)
\( u_2(x) = 1 - x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \cdots. \)

This gives the exact solution
\( u(x) = \cos x - \sin x. \)

5.6 Converting to Volterra Equations

Exercises 5.6

1. \( u''(x) = 1 + \int_0^x (x - t)u(t)dt, \ u(0) = 1, \ u'(0) = 0 \)

Integrating both sides twice from 0 to \( x, \) using the given conditions and converting the resulting multiple integrals to a single integral we obtain

\[ u(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{3!} \int_0^x (x - t)^3u(t)dt \]

Using the Adomian decomposition method, (or any other method), where we set
\[ u_0(x) = 1 + \frac{1}{2!}x^2 \]
\[ u_1(x) = \frac{1}{4!}x^4 + \frac{1}{6!}x^6 \]

Accordingly \( u(x) \) in a series form is given by
\[ u(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \]

and in a closed form
\[ u(x) = \cosh x \]

www.MathsSchoolInternational.com
Chapter 5. Volterra Integro-Differential Equations

2. \( u'(x) = 1 - \int_0^x u(t)\,dt, \quad u(0) = 0 \)

Integrating both sides from 0 to \( x \), using the given condition and converting the resulting multiple integral to a single integral we obtain

\[
u(x) = x - \int_0^x (x - t)u(t)\,dt
\]

Using the series solution method, (or any other method), where we set

\[
u(x) = a_0 + a_1x + a_2x^2 + \cdots
\]

in both sides of the resulting integral equation we find

\[
a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = \frac{1}{5!}, \quad \cdots
\]

Accordingly, \( u(x) \) in a series form is given by

\[
u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots
\]

and in a closed form

\[
u(x) = \sin x
\]

3. \( u''(x) = x + \int_0^x (x - t)u(t)\,dt, \quad u(0) = 0, \quad u'(0) = 1 \)

Integrating both sides twice from 0 to \( x \), using the given conditions and converting the resulting multiple integrals to a single integral we obtain

\[
u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7
\]

Using the successive approximation method, (or any other method), where we set

\[
u_0(x) = 0
\]

so that

\[
u_1(x) = x + \frac{1}{3!}x^3
\]

\[
u_2(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7
\]
5.6. Converting to Volterra Equations

\[ u_n(x) = \sum_{k=1}^{n} x^{2k-1} \frac{(2k-1)!}{(2k-1)!} \]

Accordingly \( u(x) \) in a closed form

\[ u(x) = \sinh x \]

4. \( u'(x) = 2 - \frac{1}{2} x^2 + \int_0^x u(t)dt, u(0) = 1 \)

Integrating both sides from 0 to \( x \), using the given condition and converting the resulting multiple integral to a single integral we obtain

\[ u(x) = 1 + 2x - \frac{1}{3!} x^3 + \int_0^x (x - t)u(t)dt \]

Using the series solution method, (or any other method), where we set

\[ u(x) = a_0 + a_1 x + a_2 x^2 + \cdots \]

in both sides of the resulting integral equation we find

\[ a_0 = 1, a_1 = 2, a_2 = \frac{1}{3!}, a_3 = \frac{1}{3!}, a_4 = \frac{1}{3!}, a_5 = \frac{1}{3!}, \cdots \]

Accordingly, \( u(x) \) in a series form is given by

\[ u(x) = x + (1 + x + \frac{1}{3!} x^2 + \frac{1}{3!} x^3 + \cdots) \]

and in a closed form

\[ u(x) = x + e^x \]

5. \( u'(x) = 1 - \int_0^x u(t)dt, u(0) = 1 \)

Integrating both sides from 0 to \( x \), using the given condition and converting the resulting multiple integral to a single integral we obtain

\[ u(x) = 1 + x - \int_0^x (x - t)u(t)dt \]

Using the series solution method, (or any other method), where we set

\[ u(x) = a_0 + a_1 x + a_2 x^2 + \cdots \]

in both sides of the resulting integral equation we find

\[ a_0 = 1, a_1 = 1, a_2 = -\frac{1}{3!}, a_3 = -\frac{1}{3!}, a_4 = \frac{1}{3!}, a_5 = \frac{1}{3!}, \cdots \]
Accordingly, \( u(x) \) in a series form is given by
\[
 u(x) = (1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots) + (x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots)
\]
and in a closed form
\[
 u(x) = \cos x + \sin x
\]

6. \( u''(x) = 1 + x + \int_0^x (x - t)u(t)dt, \ u(0) = u'(0) = 1 \)

Integrating both sides twice from 0 to \( x \), using the given conditions and converting the resulting multiple integrals to a single integral we obtain
\[
 u(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{3!} \int_0^x (x - t)^3 u(t)dt
\]
Using the Adomian decomposition method, (or any other method), where we set
\[
 u_0(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3
 u_1(x) = \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots
\]

Accordingly \( u(x) \) in a series form is given by
\[
 u(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots
\]
and in a closed form
\[
 u(x) = e^x
\]

5.7 Converting to Initial Value Problems

Exercises 5.7
1. \( u'(x) = e^x - \int_0^x u(t)dt, \ u(0) = 1 \)

Differentiating both sides with respect to \( x \) and using Leibniz rule we obtain the nonhomogeneous differential equation
\[
 u''(x) + u(x) = e^x, \ u(0) = 1, \ u'(0) = 1
\]

Solving the initial value problem we find
5.7. Converting to Initial Value Problems

\[ u(x) = A \cos x + B \sin x + \frac{1}{2} e^x \]

Using the initial conditions we find

\[ u(x) = \frac{1}{2} (\cos x + \sin x + e^x) \]

2. \[ u'(x) = 1 - \int_0^x u(t) \, dt, \quad u(0) = 0 \]

Differentiating both sides with respect to \( x \) and using Leibniz rule we obtain the homogeneous differential equation

\[ u''(x) + u(x) = 0, \quad u(0) = 0, \quad u'(0) = 1 \]

Solving the initial value problem we find

\[ u(x) = A \cos x + B \sin x \]

Using the initial conditions we find

\[ u(x) = \sin x \]

3. \[ u''(x) = -x - \frac{1}{2} x^2 + \int_0^x (x - t) u(t) \, dt, \quad u(0) = 1, \quad u'(0) = 1 \]

Differentiating both sides twice with respect to \( x \) and using Leibniz rule we obtain the nonhomogeneous differential equation

\[ u^{(iv)}(x) - u(x) = -1, \quad u(0) = u'(0) = 1, \quad u''(0) = 0, \quad u'''(0) = -1 \]

Solving the initial value problem we find

\[ u(x) = A \cos x + B \sin x + C \cosh x + D \sinh x + 1 \]

Using the initial conditions we find

\[ u(x) = 1 + \sin x \]

4. \[ u''(x) = 1 - \frac{1}{2} x^2 + \int_0^x (x - t) u(t) \, dt, \quad u(0) = 2, \quad u'(0) = 0 \]

Differentiating both sides twice with respect to \( x \) and using Leibniz rule we obtain the nonhomogeneous differential equation
Chapter 5. \textit{Volterra Integro-Differential Equations}

\begin{equation}
u^{(iv)}(x) - u(x) = -1, \ u(0) = 2, \ u^{'}(0) = 0, \ u^{''}(0) = 1, \ u^{'''}(0) = 0\end{equation}

Solving the initial value problem we find

\begin{equation}u(x) = A \cos x + B \sin x + C \cosh x + D \sinh x + 1\end{equation}

Using the initial conditions we find

\begin{equation}u(x) = 1 + \cosh x\end{equation}

5. \begin{equation}u^{''}(x) = -\frac{1}{2}x^2 - \frac{2}{3}x^3 + \int_0^x (x-t)u(t)dt, \ u(0) = 1, \ u^{'}(0) = 4\end{equation}

Differentiating both sides twice with respect to \(x\) and using Leibniz rule we obtain the nonhomogeneous differential equation

\begin{equation}u^{(iv)}(x) - u(x) = -1 - 4x, \ u(0) = 1, \ u^{'}(0) = 4, \ u^{''}(0) = u^{'''}(0) = 0\end{equation}

Solving the initial value problem we find

\begin{equation}u(x) = A \cos x + B \sin x + C \cosh x + D \sinh x + 1 + 4x\end{equation}

Using the initial conditions we find

\begin{equation}u(x) = 1 + 4x\end{equation}

6. \begin{equation}u^{''}(x) = -x - \frac{1}{3}x^2 + \int_0^x (x-t)u(t)dt, \ u(0) = \frac{1}{4}, \ u^{'}(0) = 1\end{equation}

Differentiating both sides twice with respect to \(x\) and using Leibniz rule we obtain the nonhomogeneous differential equation

\begin{equation}u^{(iv)}(x) - u(x) = -\frac{1}{4}, \ u(0) = u^{'}(0) = 1, \ u^{''}(0) = 0, \ u^{'''}(0) = -1\end{equation}

Solving the initial value problem we find

\begin{equation}u(x) = A \cos x + B \sin x + C \cosh x + D \sinh x + \frac{1}{4}\end{equation}

Using the initial conditions we find

\begin{equation}u(x) = \frac{1}{4} + \sin x\end{equation}
5.8. The Volterra Integro-Differential Equations of the First Kind

7. \( u'(x) = 1 + \sin x + \int_0^x u(t)dt, \ u(0) = -1 \)

Differentiating both sides with respect to \( x \) and using Leibniz rule we obtain the nonhomogeneous differential equation

\[
    u''(x) - u(x) = \cos x, \ u(0) = -1, \ u'(0) = 1
\]

Solving the initial value problem we find

\[
    u(x) = Ae^x + Be^{-x} - \frac{1}{2} \cos x
\]

Using the initial conditions we find

\[
    u(x) = \frac{1}{4}e^x - \frac{3}{4}e^{-x} - \frac{1}{2} \cos x
\]

5.8 The Volterra Integro-Differential Equations of the First Kind

Exercises 5.8

1. Differentiating both sides leads to

\[
    u' = \cos x - \sin x - 1 - \int_0^x u'(t) \ dt
\]

The correction functional for this equation is given by

\[
    u_{n+1}(x) = u_n(x) - \int_0^x \left( u_n'(t) - \cos t + \sin t + 1 \right) + \int_0^t u_n'(r) \ dr \ dt.
\]

We find the following successive approximations:

\[
    u_0(x) = 1,
\]

\[
    u_1(x) = \cos x + \sin x - x,
\]

\[
    u_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 + \cdots,
\]

\[
    u_3(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{6}x^6 + \cdots.
\]

This gives the exact solution
Chapter 5. Volterra Integro-Differential Equations

\[ u(x) = \cos x. \]

2. Differentiating both sides leads to

\[ u' = -1 - \int_0^x u'(t) \, dt \]

The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) + 1 + \int_0^t u'_n(r) \, dr \right) \, dt. \]

We find the following successive approximations:

\[ u_0(x) = 1, \]
\[ u_1(x) = 1 - x, \]
\[ u_2(x) = 1 - x + \frac{1}{2} x^2, \]
\[ u_3(x) = 1 - x + \frac{1}{2} x^2 - \frac{1}{3!} x^3 + \cdots, \]

This gives the exact solution

\[ u(x) = e^{-x}. \]

3. Differentiating both sides leads to

\[ u' = \sinh x + \cosh x - \int_0^x u'(t) \, dt \]

The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) + - \sinh t - \cosh t + \int_0^t u'_n(r) \, dr \right) \, dt. \]

We find the following successive approximations:

\[ u_0(x) = 0, \]
\[ u_1(x) = \cosh x + \sinh x - 1, \]
\[ u_2(x) = x, \]
\[ u_3(x) = x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots, \]
5.8. The Volterra Integro-Differential Equations of the First Kind

This gives the exact solution

\[ u(x) = \sinh x. \]

4. Differentiating both sides leads to

\[ u' = 3e^x - 2 - \int_0^x (u + u'(t)) \, dt \]

The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) - 3e^t + 2 + \int_0^t (u_n(r) + u'_n(r)) \, dr \right) \, dt. \]

We find the following successive approximations:

\[ u_0(x) = 1, \]
\[ u_1(x) = 1 + x + \frac{1}{2}x^2 + \cdots, \]
\[ u_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots, \]
\[ u_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \]

This gives the exact solution

\[ u(x) = e^x. \]

5. Differentiating both sides leads to

\[ u' = \cos x + x + \frac{1}{2}x^2 - \int_0^x (u + u'(t)) \, dt \]

The correction functional for this equation is given by

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) - \cos t - t - \frac{1}{2}t^2 + \int_0^t (u_n(r) + u'_n(r)) \, dr \right) \, dt. \]

We find the following successive approximations:

\[ u_0(x) = 1, \]
\[ u_1(x) = 1 + x + \cdots, \]
\[ u_2(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{3!}x^4 + \cdots, \]
\[ u_3(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{3!}x^4 - \frac{1}{5!}x^6 + \cdots, \]
This gives the exact solution
\[ u(x) = x + \cos x. \]

6. Differentiating both sides leads to
\[ u' = \frac{1}{2} \cosh x + \frac{3}{2} \sinh x - \frac{1}{2} - \frac{1}{2} \int_0^x (u + u'(t)) \, dt \]
The correction functional for this equation is given by
\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(t) - \cos t - t - \frac{1}{2} t^2 + \int_0^t (u_n(r) + u'_n(r)) \, dr \right) \, dt. \]

We find the following successive approximations:
\[ u_0(x) = 1, \]
\[ u_1(x) = 1 + \frac{1}{2} x^2 + \frac{1}{12} x^3 + \frac{1}{24} x^4 + \cdots, \]
\[ u_2(x) = 1 + \frac{1}{2} x^2 + \frac{1}{32} x^4 + \cdots, \]
\[ u_3(x) = 1 + \frac{1}{2} x^2 + \frac{1}{32} x^4 + \frac{1}{60} x^6 + \cdots, \]

This gives the exact solution
\[ u(x) = \cosh x. \]
Chapter 6

Singular Integral Equations

6.2 Abel’s Problem

Exercises 6.2

1. \( \pi(x + 1) = \int_0^x \frac{1}{\sqrt{x - t}} u(t) dt \)

Substituting \( f(x) = \pi(x + 1) \) into the formula

\[
u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{\sqrt{x - t}} dt
\]

we find

\[
u(x) = \frac{d}{dx} \int_0^x \frac{t + 1}{\sqrt{x - t}} dt
\]

Using the substitution \( y = x - t \) so that \( dt = -dy \) we find

\[
u(x) = \frac{d}{dx} \int_0^x \frac{x - y + 1}{y^{3/2}} dy
\]

Integrating the right hand side yields

\[
u(x) = \frac{d}{dx} \left( 2(x + 1)y^{1/2} - \frac{2}{3} y^{3/2} \right) \bigg|_{y=0}^{y=x}
\]
Chapter 6. Singular Integral Equations

\[ u(x) = 2\sqrt{x} + \frac{1}{\sqrt{x}} \]

This result can also be obtained by integrating the integral involving \( t \) by using Appendix B.

2. \( \frac{\pi}{2}(x^2 - x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \)

Substituting \( f(x) = \frac{\pi}{2}(x^2 - x) \) into the formula

\[ u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt \]

we find

\[ u(x) = \frac{1}{2} \frac{d}{dx} \int_0^x \frac{t^2 - t}{\sqrt{x-t}} dt \]

Using the substitution \( y = x - t \) so that \( dt = -dy \) we find

\[ u(x) = \frac{1}{2} \frac{d}{dx} \int_0^x \frac{(x - y)^2 - (x - y)}{y^2} dy \]

Integrating the right hand side yields

\[ u(x) = \sqrt{x} \left( \frac{4}{3} x - 1 \right) \]

This result can also be obtained by integrating the integral involving \( t \) by using Appendix B.

3. \( 1 + x + x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \)

Substituting \( f(x) = 1 + x + x^2 \) into the formula

\[ u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt \]

we find

\[ u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{1 + t + t^2}{\sqrt{x-t}} dt \]

Using the substitution \( y = x - t \) so that \( dt = -dy \) we find

\[ u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{1 + (x - y) + (x - y)^2}{y^2} dy \]
6.2. Abel’s Problem

Integrating the right hand side yields

\[ u(x) = \frac{1}{\pi \sqrt{x}} (1 + 2x + \frac{8}{3} x^2) \]

This result can also be obtained by integrating the integral involving \( t \) by using Appendix B.

4. \( \frac{3}{8} \pi x^2 = \int_0^x \frac{1}{\sqrt{x - t}} u(t) dt \)

Substituting \( f(x) = \frac{3}{8} \pi x^2 \) into the formula

\[ u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{\sqrt{x - t}} dt \]

we find

\[ u(x) = \frac{3}{8} \frac{d}{dx} \int_0^x \frac{t^2}{\sqrt{x - t}} dt \]

Using the substitution \( t = x \sin^2 \theta \) so that \( dt = 2x \sin \theta \cos \theta d\theta \) we find

\[ u(x) = x^2 \]

This result can also be obtained by integrating the integral involving \( t \) by using Appendix B.

5. \( \frac{4}{3} \pi x^2 = \int_0^x \frac{1}{\sqrt{x - t}} u(t) dt \)

Substituting \( f(x) = \frac{4}{3} \pi x^2 \) into the formula

\[ u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{\sqrt{x - t}} dt \]

we find

\[ u(x) = \frac{4}{3\pi} \frac{d}{dx} \int_0^x \frac{t^2}{\sqrt{x - t}} dt \]

Using the substitution \( t = x \sin^2 \theta \) so that \( dt = 2x \sin \theta \cos \theta d\theta \) we find

\[ u(x) = x \]

This result can also be obtained by integrating the integral involving \( t \) by using Appendix B.
Chapter 6. Singular Integral Equations

6. \[ \frac{8}{15} x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \]

Substituting \( f(x) = \frac{8}{15} x^2 \) into the formula

\[ u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{\sqrt{x-t}} dt \]

we find

\[ u(x) = \frac{8}{15\pi} \frac{d}{dx} \int_0^x \frac{t^2}{\sqrt{x-t}} dt \]

Using the substitution \( t = x \sin^2 \theta \) so that \( dt = 2x \sin \theta \cos \theta d\theta \) we find

\[ u(x) = \frac{1}{2} x^2 \]

This result can also be obtained by integrating the integral involving \( t \) by using Appendix B.

7. \[ x^3 = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \]

Substituting \( f(x) = x^3, f(0) = 0, f'(x) = 3x^2, \alpha = \frac{1}{2}, \sin(\alpha \pi) = 1 \) into the formula of the generalized Abel’s integral equation

\[ u(x) = \frac{1}{\pi} \int_0^x \frac{f'(t)}{\sqrt{x-t}} dt \]

we find

\[ u(x) = \frac{1}{\pi} \int_0^x \frac{3t^2}{\sqrt{x-t}} dt \]

Using the substitution \( y = x - t \) so that \( dt = -dy \) or using Appendix B we find

\[ u(x) = \frac{16}{5\pi} x^2 \]

8. \[ x^4 = \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt \]

Substituting \( f(x) = x^4, f(0) = 0, f'(x) = 4x^3, \alpha = \frac{1}{2}, \sin(\alpha \pi) = 1 \) into the formula of the generalized Abel’s integral equation
6.2. Abel’s Problem

\[ u(x) = \frac{1}{\pi} \int_0^x \frac{f'(t)}{\sqrt{x-t}} \, dt \]

we find

\[ u(x) = \frac{1}{\pi} \int_0^x \frac{4t^3}{\sqrt{x-t}} \, dt \]

Using the substitution \( y = x - t \) so that \( dt = -dy \) or using Appendix B we find

\[ u(x) = \frac{128}{35\pi} x^\frac{7}{2} \]

9. \( x + x^3 = \int_0^x \frac{1}{\sqrt{x-t}} u(t) \, dt \)

Substituting \( f(x) = x + x^3, f(0) = 0, f'(x) = 1 + 3x^2, \alpha = \frac{1}{2}, \sin(\alpha \pi) = 1 \) into the formula of the generalized Abel’s integral equation

\[ u(x) = \frac{1}{\pi} \int_0^x \frac{f'(t)}{\sqrt{x-t}} \, dt \]

we find

\[ u(x) = \frac{1}{\pi} \int_0^x \frac{1 + 3t^2}{\sqrt{x-t}} \, dt \]

Using the substitution \( y = x - t \) so that \( dt = -dy \) or using Appendix B we find

\[ u(x) = \frac{2\sqrt{x}}{\pi} (1 + \frac{8}{5} x^2) \]

10. \( \sin x = \int_0^x \frac{1}{\sqrt{x-t}} u(t) \, dt \)

Substituting \( f(x) = \sin x, f(0) = 0, f'(x) = \cos x \) into the formula of the generalized equation

\[ u(x) = \frac{1}{\pi} \int_0^x \frac{f'(t)}{\sqrt{x-t}} \, dt \]

Using the approximation \( \cos x \approx 1 \), we find

\[ u(x) \approx \frac{2}{\pi} \sqrt{x} \]
6.3 Generalized Abel’s Problem

Exercises 6.3

1. Notice that $\alpha = \frac{1}{6}, f(x) = \frac{36}{55}x^{\frac{11}{6}}$. Therefore, we set
   \[ u(x) = \sin(\alpha \pi) \frac{d}{dx} \int_0^x \frac{36}{55} \frac{11}{(x-t)^\frac{11}{6}} \, dt = x. \]

2. Notice that $\alpha = \frac{1}{3}, f(x) = \frac{243}{440}x^{\frac{11}{3}}$. Therefore, we set
   \[ u(x) = \sin(\alpha \pi) \frac{d}{dx} \int_0^x \frac{243}{440} \frac{11}{(x-t)^\frac{11}{3}} \, dt = x^3. \]

3. Notice that $\alpha = \frac{3}{4}, f(x) = 24x^{\frac{7}{4}}$. Therefore, we set
   \[ u(x) = \sin(\alpha \pi) \frac{d}{dx} \int_0^x \frac{24}{x^{\frac{7}{4}}} \, dt = 6. \]

4. Notice that $\alpha = \frac{2}{3}, f(x) = 3\pi x^\frac{1}{3} + 9x^\frac{4}{3}, u(x) = \pi + 4x$.

5. Notice that $\alpha = \frac{5}{6}, f(x) = \frac{36}{7}x^\frac{7}{6}, u(x) = x$.

6. Notice that $\alpha = \frac{2}{3}, f(x) = 27x^\frac{7}{3} + 9x^\frac{4}{3}, u(x) = 4x + 14x^2$.

6.4 The Weakly Singular Volterra Equations

Exercises 6.4

1. $u(x) = \sqrt{x} - \pi x + 2 \int_0^x \frac{1}{\sqrt{x-t}} \, dt, \ I = [0, 2]\]

   Using the Adomian decomposition method, we substitute
   \[ u(x) = \sum_{n=0}^\infty u_n(x) \]
   into both sides of the integral equation, hence we obtain
   \[ u_0(x) + u_1(x) + u_2(x) + \cdots = \sqrt{x} - \pi x \]
   \[ + 2 \int_0^x \frac{1}{\sqrt{x-t}} \left( u_0(t) + u_1(t) + \cdots \right) \, dt \]

   Setting the zeroth component by

   $\sum_{n=0}^\infty u_n(x) = \frac{\sqrt{x} - \pi x}{2} + \int_0^x \frac{1}{\sqrt{x-t}} \left( u_0(t) + u_1(t) + \cdots \right) \, dt$
6.4. The Weakly Singular Volterra Equations

\[ u_0(x) = \sqrt{x} - \pi x \]

yields the first component

\[ u_1(x) = 2 \int_0^x \frac{1}{\sqrt{x-t}} (\sqrt{t} - \pi t) dt \]

so that

\[ u_1(x) = \pi x - \frac{8}{3} \pi^2 x^{3/2} \]

by using Appendix B.

Canceling the noise term between \( u_0(x) \) and \( u_1(x) \) yields the exact solution

\[ u(x) = \sqrt{x} \]

2. \( u(x) = \frac{3}{8} \pi x^2 + x^{3/2} - \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt, \ I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[ u(x) = \sum_{n=0}^\infty u_n(x) \]

into both sides of the integral equation, hence we obtain

\[ u_0(x) + u_1(x) + u_2(x) + \cdots = \frac{3}{8} \pi x^2 + x^{3/2} \]

\[ - \int_0^x \frac{1}{\sqrt{x-t}} (u_0(t) + u_1(t) + \cdots) dt \]

Setting the zeroth component by

\[ u_0(x) = \frac{3}{8} \pi x^2 + x^{3/2} \]

yields the first component

\[ u_1(x) = - \int_0^x \frac{1}{\sqrt{x-t}} (t^{3/2} - \frac{3}{8\pi} t^2) dt \]

so that
Chapter 6. Singular Integral Equations

\[ u_1(x) = -\frac{3}{8}\pi x^2 - \frac{8}{120\pi}x^{5/2} \]

Canceling the noise term between \( u_0(x) \) and \( u_1(x) \) yields the exact solution

\[ u(x) = x^{3/2} \]

3. \( u(x) = \frac{1}{2} - \sqrt{x} + \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt, \ I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \]

into both sides of the integral equation, hence we obtain

\[ u_0(x) + u_1(x) + u_2(x) + \cdots = \frac{1}{2} - \sqrt{x} + \int_0^x \frac{1}{\sqrt{x-t}} (u_0(t) + u_1(t) + \cdots) dt \]

Setting the zeroth component by

\[ u_0(x) = \frac{1}{2} - \sqrt{x} \]

yields the first component

\[ u_1(x) = \sqrt{x} - \frac{1}{2}\pi x \]

Canceling the noise term between \( u_0(x) \) and \( u_1(x) \) yields the exact solution

\[ u(x) = \frac{1}{2} \]

4. \( u(x) = \sqrt{x} - \frac{1}{2}\pi x + \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt, \ I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \]

into both sides of the integral equation, hence we obtain

\[ u_0(x) + u_1(x) + u_2(x) + \cdots = \sqrt{x} - \frac{1}{2}\pi x \]
6.4. The Weakly Singular Volterra Equations

\[ + \int_0^x \frac{1}{\sqrt{x-t}} (u_0(t) + u_1(t) + \cdots) \, dt \]

Setting the zeroth component by

\[ u_0(x) = \sqrt{x} - \frac{1}{2} \pi x \]

yields the first component

\[ u_1(x) = \frac{1}{2} \pi x - \frac{2}{3} \pi x^{3/2} \]

Canceling the noise term between \( u_0(x) \) and \( u_1(x) \) yields the exact solution

\[ u(x) = \sqrt{x} \]

5. \( u(x) = x^{5/2} - \frac{5}{16} \pi x^3 + \int_0^x \frac{1}{\sqrt{x-t}} u(t) \, dt, \quad I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \]

into both sides of the integral equation, hence we obtain

\[ u_0(x) + u_1(x) + u_2(x) + \cdots = x^{5/2} - \frac{5}{16} \pi x^3 \]

\[ + \int_0^x \frac{1}{\sqrt{x-t}} (u_0(t) + u_1(t) + \cdots) \, dt \]

Setting the zeroth component by

\[ u_0(x) = x^{5/2} - \frac{5}{16} \pi x^3 \]

yields the first component

\[ u_1(x) = \frac{5}{16} \pi x^3 - \frac{2}{7} \pi x^{7/2} \]

Canceling the noise term between \( u_0(x) \) and \( u_1(x) \) yields the exact solution

\[ u(x) = x^{5/2} \]
Chapter 6. Singular Integral Equations

6. \( u(x) = x^3 + \frac{32}{33}x^{7/2} - \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt, \ I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[
u(x) = \sum_{n=0}^{\infty} u_n(x)\]

into both sides of the integral equation, hence we obtain

\[
u_0(x) + \nu_1(x) + \nu_2(x) + \cdots = x^3 + \frac{32}{33}x^{7/2} - \int_0^x \frac{1}{\sqrt{x-t}}(u_0(t) + u_1(t) + \cdots) dt\]

Setting the zeroth component by

\[
u_0(x) = x^3 + \frac{32}{33}x^{7/2}\]

yields the first component

\[
u_1(x) = -\frac{32}{33}x^{7/2} - \frac{25}{112}\pi x^4\]

Canceling the noise term between \( \nu_0(x) \) and \( \nu_1(x) \) yields the exact solution

\[u(x) = x^3\]

7. \( u(x) = 1 + x - 2\sqrt{x} - \frac{4}{3}x^{3/2} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt, \ I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[
u(x) = \sum_{n=0}^{\infty} u_n(x)\]

into both sides of the integral equation, hence we obtain

\[
u_0(x) + \nu_1(x) + \nu_2(x) + \cdots = 1 + x - 2\sqrt{x} - \frac{4}{3}x^{3/2} + \int_0^x \frac{1}{\sqrt{x-t}}(u_0(t) + u_1(t) + \cdots) dt\]

Setting the zeroth component by
6.4. The Weakly Singular Volterra Equations

\[ u_0(x) = 1 + x - 2\sqrt{x} - \frac{4}{3}x^{3/2} \]

yields the first component

\[ u_1(x) = 2\sqrt{x} + \frac{4}{3}x^{3/2} - \pi x - \frac{1}{2}\pi x^2 \]

Canceling the noise terms between \( u_0(x) \) and \( u_1(x) \) yields the exact solution

\[ u(x) = 1 + x \]

8. \( u(x) = 1 + 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}} u(t) \, dt, \quad I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \]

into both sides of the integral equation, hence we obtain

\[ u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + 2\sqrt{x} \]

\[ - \int_0^x \frac{1}{\sqrt{x-t}} (u_0(t) + u_1(t) + \cdots) \, dt \]

Setting the zeroth component by

\[ u_0(x) = 1 + 2\sqrt{x} \]

yields the first component

\[ u_1(x) = -2\sqrt{x} - \pi x \]

Canceling the noise terms between \( u_0(x) \) and \( u_1(x) \) yields the exact solution

\[ u(x) = 1 \]

9. \( u(x) = x^2 + \frac{16}{15}x^3 - \int_0^x \frac{1}{\sqrt{x-t}} u(t) \, dt, \quad I = [0, 2] \)

Using the Adomian decomposition method, we substitute

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \]
into both sides of the integral equation, hence we obtain

$$u_0(x) + u_1(x) + u_2(x) + \cdots = x^2 + \frac{16}{15} x^2$$

$$- \int_0^x \frac{1}{\sqrt{x-t}} (u_0(t) + u_1(t) + \cdots) \, dt$$

Setting the zeroth component by

$$u_0(x) = x^2 + \frac{16}{15} x^2$$

yields the first component

$$u_1(x) = -\frac{16}{15} x^{5/2} - \frac{1}{3} \pi x^3$$

Canceling the noise term between $u_0(x)$ and $u_1(x)$ yields the exact solution

$$u(x) = x^2$$

10. $u(x) = \frac{2}{5} \sqrt{x} + \frac{15}{16} x^2 - x - x^{5/2} + \int_0^x \frac{1}{\sqrt{x-t}} u(t) \, dt$, $I = [0, 2]$}

Using the Adomian decomposition method, we substitute

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

into both sides of the integral equation, hence we obtain

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \frac{2}{5} \sqrt{x} + \frac{15}{16} x^2 - x - x^{5/2}$$

$$+ \int_0^x \frac{1}{\sqrt{x-t}} (u_0(t) + u_1(t) + \cdots) \, dt$$

Setting the zeroth component by

$$u_0(x) = \frac{2}{5} \sqrt{x} + \frac{15}{16} x^2 - x - x^{5/2}$$

yields the first component

$$u_1(x) = x + x^{5/2} - \frac{4}{3} x^{3/2} - \frac{5}{16} \pi x^3$$

Canceling the noise terms between $u_0(x)$ and $u_1(x)$ yields the exact solution
6.4. The Weakly Singular Volterra Equations

\[ u(x) = \frac{2}{\pi} \sqrt{x} + \frac{15}{16} x^2 \]

11. We use the recurrence relation

\[ u_0(x) = 1 + x, \]

\[ u_1(x) = -\frac{4}{3} x^{\frac{3}{2}} - \frac{16}{21} x^2 + \int_0^x \frac{1+t}{(x-t)^{\frac{5}{2}}} u(t) \, dt = 0. \]

This gives the exact solution

\[ u(x) = 1 + x. \]

12. We use the recurrence relation

\[ u_0(x) = x + x^2, \]

\[ u_1(x) = -\frac{9}{4} x^{\frac{5}{2}} - \frac{243}{140} x^3 + \int_0^x \frac{t+t^2}{(x-t)^{\frac{7}{2}}} u(t) \, dt = 0. \]

This gives the exact solution

\[ u(x) = x + x^2. \]

13. We use the recurrence relation

\[ u_0(x) = 1 + 3x^2. \]

Proceed as before, to get the exact solution

\[ u(x) = 1 + 3x^2. \]

14. We use the recurrence relation

\[ u_0(x) = 5 - x. \]

Proceed as before, to get the exact solution

\[ u(x) = 5 - x. \]
Chapter 6. Singular Integral Equations

6.5 The Weakly Singular Fredholm Equations

Exercises 6.5

1. We first decompose \( f(x) \) into two parts as

\[
f_0(x) = x^2
\]
\[
f_1(x) = -\frac{16}{15} x^{\frac{7}{2}} + \frac{2}{5} \sqrt{x-1} + \frac{8 x \sqrt{x-1}}{15} + \frac{16 x^2 \sqrt{x-1}}{15}
\]

We then use the modified recurrence relation

\[
u_0(x) = x^2,
\]
\[
u_1(x) = -\frac{16}{15} x^{\frac{7}{2}} + \frac{2}{5} \sqrt{x-1} + \frac{8 x \sqrt{x-1}}{15} + \frac{16 x^2 \sqrt{x-1}}{15} + \int_1^x \frac{u_0(t)}{\sqrt{x-t}} u_0(t) \, dt = 0.
\]

This gives the exact solution by

\[u(x) = x^2.\]

2. We first decompose \( f(x) \) into two parts as

\[
f_0(x) = 10
\]
\[
f_1(x) = 20(x - 1)^{\frac{1}{2}} - 20(x + 1)^{\frac{1}{2}}
\]

We then use the modified recurrence relation

\[
u_0(x) = 10,
\]
\[
u_1(x) = 20(x - 1)^{\frac{1}{2}} - 20(x + 1)^{\frac{1}{2}} + \int_{-1}^1 \frac{u_0(t)}{(x-t)^{\frac{1}{2}}} \, dt = 0.
\]

This gives the exact solution by

\[u(x) = 10.\]

3. We first decompose \( f(x) \) into two parts as

\[
f_0(x) = 10x
\]
\[
f_1(x) = -9x^{\frac{7}{2}} + 6(x - 1)^{\frac{1}{2}} + 9x(x - 1)^{\frac{3}{2}}
\]

We then use the modified recurrence relation
6.5. The Weakly Singular Fredholm Equations

\[ u_0(x) = 10x, \]
\[ u_1(x) = -9x^{\frac{7}{3}} + 6(x-1)^{\frac{4}{3}} + 9x(x-1)^{\frac{2}{3}} + \int_0^1 \frac{u(t)}{(x-t)^{\frac{2}{3}}} u_0(t) \, dt = 0. \]

This gives the exact solution by \( u(x) = 10x. \)

4. We first decompose \( f(x) \) into two parts as

\[
\begin{align*}
f_0(x) &= 3 + 10x - 9x^\frac{1}{3} \\
f_1(x) &= -\frac{45}{2} x^{\frac{4}{3}} + \frac{33}{2} (x-1)^{\frac{4}{3}} + \frac{45}{2} x(x-1)^{\frac{2}{3}}
\end{align*}
\]

We then use the modified recurrence relation

\[
\begin{align*}
u_0(x) &= 3 + 10x - 9x^\frac{1}{3}, \\
u_1(x) &= -\frac{45}{2} x^{\frac{4}{3}} + \frac{33}{2} (x-1)^{\frac{4}{3}} + \frac{45}{2} x(x-1)^{\frac{2}{3}} + \int_0^1 \frac{u(t)}{(x-t)^{\frac{2}{3}}} \, dt = 9x^{\frac{1}{3}} + \cdots.
\end{align*}
\]

Cancelling the noise term gives the exact solution by \( u(x) = 3 + 10x. \)

5. We first decompose \( f(x) \) into two parts as

\[
\begin{align*}
f_0(x) &= x + x^2 - \frac{16}{5} x^{\frac{4}{3}} (1 + \frac{8}{3} x) \\
f_1(x) &= \frac{5}{45} (x-1)^{\frac{1}{3}} (7 + 22x + 16x^2)
\end{align*}
\]

We then use the modified recurrence relation

\[
\begin{align*}
u_0(x) &= x + x^2 - \frac{16}{5} x^{\frac{4}{3}} (1 + \frac{8}{3} x), \\
u_1(x) &= \frac{5}{45} (x-1)^{\frac{1}{3}} (7 + 22x + 16x^2) + \int_0^1 \frac{u(t)}{(x-t)^{\frac{2}{3}}} \, dt = \frac{16}{5} x^{\frac{2}{3}} (1 + \frac{8}{3} x) + \cdots.
\end{align*}
\]

Cancelling the noise term gives the exact solution by \( u(x) = x + x^2. \)
Chapter 7
Nonlinear Fredholm Integral Equations

7.2 Nonlinear Fredholm Integral Equations

7.2.1 The Direct Computation Method

Exercises  7.2.1

1. \( u(x) = 1 + \frac{1}{2} \lambda \int_{0}^{1} u^2(t) dt \)

This equation can be rewritten as

\[ u(x) = 1 + \frac{1}{2} \lambda \alpha \]

where

\[ \alpha = \int_{0}^{1} u^2(t) dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_{0}^{1} \left(1 + \frac{1}{2} \lambda \alpha \right)^2 dt \]

Integrating the definite integral yields

\[ \lambda^2 \alpha^2 + (4\lambda - 4)\alpha + 4 = 0 \]

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Chapter 7. Nonlinear Fredholm Integral Equations

Solving the quadratic equation for $\alpha$ gives

$$\alpha = \frac{2(1 - \lambda) \pm 2\sqrt{1 - 2\lambda}}{2\lambda^2}$$

Substituting for $\alpha$ in the above equation for $u(x)$ we find

$$u(x) = \frac{1 \pm \sqrt{1 - 2\lambda}}{\lambda}$$

Examining the solution $u(x)$ we obtained leads to

- $\lambda = 0$ is a singular point
- $\lambda = \frac{1}{2}$ is a bifurcation point

2. $u(x) = 1 - \lambda \int_0^1 u^2(t)dt$

This equation can be rewritten as

$$u(x) = 1 - \lambda \alpha$$

where

$$\alpha = \int_0^1 u^2(t)dt$$

Substituting for $u(x)$ from the above equation we obtain

$$\alpha = \int_0^1 (1 - \lambda \alpha)^2 dt$$

Integrating the definite integral yields

$$\lambda^2 \alpha^2 - (1 + 2\lambda)\alpha + 1 = 0$$

Solving the quadratic equation for $\alpha$ gives

$$\alpha = \frac{(1 + 2\lambda) \pm 2\sqrt{1 + 4\lambda}}{2\lambda^2}$$

Substituting for $\alpha$ in the above equation for $u(x)$ we find

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7.2. Nonlinear Fredholm Integral Equations

\[ u(x) = \frac{1 \pm \sqrt{1 + 4\lambda}}{2\lambda} \]

Examining the solution \( u(x) \) we obtained leads to

\[ \lambda = 0 \] is a singular point

\[ \lambda = -\frac{1}{4} \] is a bifurcation point

3. \( u(x) = 1 + \lambda \int_0^1 tu^2(t)dt \)

This equation can be rewritten as

\[ u(x) = 1 + \lambda \alpha \]

where

\[ \alpha = \int_0^1 tu^2(t)dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 t (1 + \lambda \alpha)^2 dt \]

Integrating the definite integral yields

\[ \lambda^2 \alpha^2 + (2\lambda - 2)\alpha + 1 = 0 \]

Solving the quadratic equation for \( \alpha \) gives

\[ \alpha = \frac{(1 - \lambda) \pm \sqrt{1 - 2\lambda}}{\lambda^2} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = \frac{1 \pm \sqrt{1 - 2\lambda}}{\lambda} \]

Examining the solution \( u(x) \) we obtained leads to

\[ \lambda = 0 \] is a singular point
4. \( u(x) = 1 + \lambda \int_0^1 t^2 u^2(t) dt \)

This equation can be rewritten as

\[ u(x) = 1 + \lambda \alpha \]

where

\[ \alpha = \int_0^1 t^2 u^2(t) dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 t^2 (1 + \lambda \alpha)^2 dt \]

Integrating the definite integral yields

\[ \lambda^2 \alpha^2 + (2\lambda - 3)\alpha + 1 = 0 \]

Solving the quadratic equation for \( \alpha \) gives

\[ \alpha = \frac{(3 - 2\lambda) \pm \sqrt{9 - 12\lambda}}{2\lambda^2} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = \frac{3 \pm \sqrt{9 - 12\lambda}}{2\lambda} \]

Examining the solution \( u(x) \) we obtained leads to the conclusion that

\( \lambda = 0 \) is a singular point

\( \lambda = \frac{3}{4} \) is a bifurcation point

5. \( u(x) = 1 + \lambda \int_0^1 t^3 u^2(t) dt \)

This equation can be rewritten as
7.2. Nonlinear Fredholm Integral Equations

\[ u(x) = 1 + \lambda \alpha \]

where

\[ \alpha = \int_0^1 t^3 u^2(t) dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 t^3 (1 + \lambda \alpha)^2 \, dt \]

Integrating the definite integral yields

\[ \lambda^2 \alpha^2 + (2\lambda - 4)\alpha + 1 = 0 \]

Solving the quadratic equation for \( \alpha \) gives

\[ \alpha = \frac{(2 - \lambda) \pm 2\sqrt{1 - \lambda}}{\lambda^2} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = \frac{2 \pm 2\sqrt{1 - \lambda}}{\lambda} \]

Examining the solution \( u(x) \) we obtained leads to

- \( \lambda = 0 \) is a singular point
- \( \lambda = 1 \) is a bifurcation point

6. \( u(x) = 2 - \frac{4}{3} x + \int_0^1 x t^2 u^2(t) \, dt \)

This equation can be rewritten as

\[ u(x) = 2 + (\alpha - \frac{4}{3}) x \]

where

\[ \alpha = \int_0^1 t^2 u^2(t) \, dt \]

Substituting for \( u(x) \) from the above equation we obtain
Chapter 7. Nonlinear Fredholm Integral Equations

\[ \alpha = \int_0^1 t^2 \left( 2 + \left( \alpha - \frac{4}{3} t \right) \right)^2 dt \]

Integrating the definite integral yields

\[ \alpha = \frac{4}{3} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = 2 \]

7. \( u(x) = \sin x - \frac{\pi}{8} + \frac{1}{2} \int_0^{\pi/2} u^2(t) dt \)

This equation can be rewritten as

\[ u(x) = \sin x + \left( \frac{1}{2} \alpha - \frac{\pi}{8} \right) \]

where

\[ \alpha = \int_0^1 u^2(t) dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 \left( \sin t + \left( \frac{1}{2} \alpha - \frac{\pi}{8} \right) \right)^2 dt \]

Integrating the definite integral yields

\[ \alpha = \frac{\pi}{4} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = \sin x \]

8. \( u(x) = \cos x - \frac{\pi}{8} + \frac{1}{2} \int_0^{\pi/2} u^2(t) dt \)

This equation can be rewritten as

\[ u(x) = \cos x + \left( \frac{1}{2} \alpha - \frac{\pi}{8} \right) \]
7.2. Nonlinear Fredholm Integral Equations

where

\[ \alpha = \int_0^1 u^2(t) \, dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 \left( \cos t + \left( \frac{1}{2} \alpha - \frac{\pi}{8} \right) \right)^2 \, dt \]

Integrating the definite integral yields

\[ \alpha = \frac{\pi}{4} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = \cos x \]

9. \( u(x) = x - \frac{1}{8} + \frac{1}{2} \int_0^1 t u^2(t) \, dt \)

This equation can be rewritten as

\[ u(x) = x + \left( \frac{1}{2} \alpha - \frac{1}{8} \right) \]

where

\[ \alpha = \int_0^1 t u^2(t) \, dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 t \left( t + \left( \frac{1}{2} \alpha - \frac{1}{8} \right) \right)^2 \, dt \]

Integrating the definite integral yields

\[ \alpha = \frac{1}{4}, \frac{67}{12} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = x, x + \frac{8}{3} \]

10. \( u(x) = x^2 - \frac{1}{m} + \frac{1}{2} \int_0^1 u^2(t) \, dt \)

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Chapter 7. Nonlinear Fredholm Integral Equations

This equation can be rewritten as

\[ u(x) = x^2 + \left( \frac{1}{2} \alpha - \frac{1}{10} \right) \]

where

\[ \alpha = \int_0^1 u^2(t)dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 \left( t^2 + \left( \frac{1}{2} \alpha - \frac{1}{10} \right)^2 \right) dt \]

Integrating the definite integral yields

\[ \alpha = \frac{1}{5}, \frac{43}{15} \]

Substituting for \( \alpha \) in the above equation for \( u(x) \) we find

\[ u(x) = x, x^2 + \frac{4}{3} \]

11. \( u(x) = x - \frac{5}{6} + \int_0^1 (u(t) + u^2(t)) dt \)

This equation can be rewritten as

\[ u(x) = x + (\alpha - \frac{5}{6}) \]

where

\[ \alpha = \int_0^1 (u(t) + u^2(t)) dt \]

Substituting for \( u(x) \) from the above equation we obtain

\[ \alpha = \int_0^1 \left( t + (\alpha - \frac{5}{6}) + (t + (\alpha - \frac{5}{6}))^2 \right) dt \]

Integrating the definite integral yields

\[ \alpha = \frac{5}{6}, -\frac{1}{6} \]
7.2. Nonlinear Fredholm Integral Equations

Substituting for $\alpha$ in the above equation for $u(x)$ we find

$$u(x) = x, x - 1$$

12. $u(x) = x - 1 + \frac{3}{4} \int_0^1 (2t + u^2(t)) \, dt$

This equation can be rewritten as

$$u(x) = x + (\frac{3}{4} \alpha - 1)$$

where

$$\alpha = \int_0^1 (2t + u^2(t)) \, dt$$

Substituting for $u(x)$ from the above equation we obtain

$$\alpha = \int_0^1 \left(2t + (t + (\frac{3}{4} \alpha - 1))^2\right) \, dt$$

Integrating the definite integral yields

$$\alpha = \frac{4}{3}, \frac{16}{9}$$

Substituting for $\alpha$ in the above equation for $u(x)$ we find

$$u(x) = x, x + \frac{1}{3}$$

7.2.2 The Adomian Decomposition Method

Exercises 7.2.2

1. $u(x) = 1 + \lambda \int_0^1 tu^2(t) \, dt, \lambda \leq \frac{1}{2}$

Using the decomposition $u(x) = \sum_{n=0}^{\infty} u_n(x)$ into both sides and following the procedure of the Adomian polynomials discussed before, we set

$$u_0(x) = 1$$

Accordingly, the first components is given by
Chapter 7. Nonlinear Fredholm Integral Equations

\[ u_1(x) = \lambda \int_0^1 t \, dt = \frac{1}{2} \lambda \]

\[ u_2(x) = \lambda \int_0^1 t(2u_0(t)u_1(t)) \, dt = \frac{1}{2} \lambda^2 \]

\[ u_3(x) = \frac{5}{8} \lambda^3 \]

Consequently the solution in a series form is

\[ u(x) = 1 + \frac{1}{2} \lambda + \frac{1}{2} \lambda^2 + \frac{5}{8} \lambda^3 + \ldots \]

2. \( u(x) = 1 + \lambda \int_0^1 t^3 u^2(t) \, dt, \ \lambda \leq 1 \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the procedure of the Adomian polynomials discussed before, we set

\[ u_0(x) = 1 \]

Accordingly, the first components is given by

\[ u_1(x) = \lambda \int_0^1 t^3 \, dt = \frac{1}{4} \lambda \]

\[ u_2(x) = \lambda \int_0^1 t^3 (2u_0(t)u_1(t)) \, dt = \frac{1}{8} \lambda^2 \]

Consequently the solution in a series form is

\[ u(x) = 1 + \frac{1}{4} \lambda + \frac{1}{8} \lambda^2 + \ldots \]

3. \( u(x) = 2 \sin x - \frac{x}{8} + \frac{1}{8} \int_0^{\pi/2} u^2(t) \, dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = 2 \sin x \]

Accordingly, the first components is given by

\[ u_1(x) = -\frac{\pi}{8} + \frac{1}{8} \int_0^{\pi/2} 4\sin^2 t \, dt = 0 \]
7.2. Nonlinear Fredholm Integral Equations

\[ u_2(x) = 0 \]
\[ u_3(x) = 0 \]

Consequently the solution is

\[ u(x) = 2 \sin x \]

4. \( u(x) = 2 \cos x - \frac{\pi}{8} + \frac{1}{8} \int_0^{\pi/2} u^2(t)dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = 2 \cos x \]

Accordingly, the first component is given by

\[ u_1(x) = -\frac{\pi}{8} + \frac{1}{8} \int_0^{\pi/2} 4 \cos^2 t dt = 0 \]
\[ u_2(x) = 0 \]
\[ u_3(x) = 0 \]

Consequently the solution is

\[ u(x) = 2 \cos x \]

5. \( u(x) = \sec x - x + x \int_0^{\pi/4} u^2(t)dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = \sec x \]

Accordingly, the first component is given by

\[ u_1(x) = -x + x \int_0^{\pi/4} \sec^2 t dt = 0 \]
\[ u_2(x) = 0 \]
Chapter 7. Nonlinear Fredholm Integral Equations

\[ u_3(x) = 0 \]

Consequently the solution is

\[ u(x) = \sec x \]

6. \( u(x) = \frac{3}{2}x + \frac{3}{8} \int_0^1 xu^2(t)dt \)

Using the decomposition \( u(x) = \sum_{n=0}^\infty u_n(x) \) into both sides and following the procedure of the Adomian polynomials discussed before, we set

\[ u_0(x) = \frac{3}{2}x \]

Accordingly, the first component is given by

\[ u_1(x) = \frac{9}{32}x \]

\[ u_2(x) = \frac{27}{256}x \]

Consequently the solution in a series form is given by

\[ u(x) = \frac{3}{2}x + \frac{9}{32}x + \frac{27}{256}x + \cdots \]

so that

\[ u(x) \approx 2x \]

7. \( u(x) = x^2 - \frac{1}{12} + \frac{1}{2} \int_0^1 tu^2(t)dt \)

Using the decomposition \( u(x) = \sum_{n=0}^\infty u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = x^2 \]

Accordingly, the first components is given by

\[ u_1(x) = -\frac{1}{12} + \frac{1}{2} \int_0^1 t^5 dt = 0 \]

\[ u_2(x) = 0 \]

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7.2. **Nonlinear Fredholm Integral Equations**

Consequently the solution is given by

\[ u(x) = x^2 \]

8. \[ u(x) = x - \frac{\pi}{8} + \frac{1}{2} \int_0^1 \frac{1}{1 + u^2(t)} \, dt \]

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = x \]

Accordingly, the first component is given by

\[ u_1(x) = -\frac{\pi}{8} + \frac{1}{2} \int_0^1 \frac{1}{1 + t^2} \, dt = 0 \]
\[ u_2(x) = 0 \]

Consequently the solution is given by

\[ u(x) = x \]

9. \[ u(x) = x - 1 + \frac{2}{\pi} \int_{-1}^1 \frac{1}{1 + u^2(t)} \, dt \]

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = x \]

Accordingly, the first component is given by

\[ u_1(x) = -1 + \frac{2}{\pi} \int_{-1}^1 \frac{1}{1 + t^2} \, dt = 0 \]

Consequently the solution, after justifying that \( u_0(x) \) justifies the equation, is given by

\[ u(x) = x \]

10. \[ u(x) = x - \frac{1}{4} \ln 2 + \frac{1}{2} \int_0^1 \frac{t}{1 + u^2(t)} \, dt \]
Chapter 7. Nonlinear Fredholm Integral Equations

Using the decomposition \(u(x) = \sum_{n=0}^{\infty} u_n(x)\) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = x \]

Accordingly, the first component is given by

\[ u_1(x) = -\frac{1}{4} \ln 2 + \frac{1}{2} \int_0^1 \frac{t}{1 + t^2} dt, = 0 \]

Consequently the solution, after justifying that \(u_0(x)\) justifies the equation, is given by

\[ u(x) = x \]

11. \(u(x) = \sin x + \cos x - \frac{\pi + 2}{8} + \frac{1}{4} \int_0^{\pi/2} u^2(t) dt\)

Using the decomposition \(u(x) = \sum_{n=0}^{\infty} u_n(x)\) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = \sin x + \cos x \]

Accordingly, the first component is given by

\[ u_1(x) = -\frac{\pi + 2}{8} + \frac{1}{4} \int_0^{\pi/2} (1 + \sin 2t) dt - 0. \]

Consequently the solution, after justifying that \(u_0(x)\) justifies the equation, is given by

\[ u(x) = \sin x + \cos x \]

12. \(u(x) = \sinh x - 1 + \int_0^1 (\cosh^2 t - u^2(t)) dt\)

Using the decomposition \(u(x) = \sum_{n=0}^{\infty} u_n(x)\) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = \sinh x \]

Accordingly, the first component is given by
7.2. Nonlinear Fredholm Integral Equations

\[ u_1(x) = 0 \]

Consequently the solution, after justifying that \( u_0(x) \) justifies the equation, is given by

\[ u(x) = \sinh x \]

13. \( u(x) = \cos x + 2 - \int_0^1 (1 + \sin^2 t + u^2(t)) \, dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = \cos x \]

Accordingly, the first component is given by

\[ u_1(x) = 0 \]

Consequently the solution, after justifying that \( u_0(x) \) justifies the equation, is given by

\[ u(x) = \cos x \]

14. \( u(x) = \sec x - x + \int_0^1 x \left( u^2(t) - \tan^2 t \right) \, dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the modified decomposition method discussed before, we set

\[ u_0(x) = \sec x \]

Accordingly, the first component is given by

\[ u_1(x) = 0 \]

Consequently the solution, after justifying that \( u_0(x) \) justifies the equation, is given by

\[ u(x) = \sec x \]
Chapter 7. Nonlinear Fredholm Integral Equations

7.2.3 The Variational Iteration Method

Exercises 7.2.3

1. \( u(x) = \frac{3}{4}x + \int_0^1 xt^2(t) \, dt \)

Differentiating both sides of this equation with respect to \( x \) yields

\[
\frac{du}{dx} = \frac{3}{4} + \int_0^1 tu^2(t) \, dt, \quad u(0) = 0.
\]

The correction functional for this equation is given by

\[
u_{n+1}(x) = u_n(x) - \int_0^x \left( \frac{u_n'(t) - \frac{3}{4} - \int_0^1 ru_n^2(r) \, dr}{dt} \right) \, dt.
\]

where we used \( \lambda = -1 \) for first-order integro-differential equations. This gives the following successive approximations

\[
\begin{align*}
u_0(x) &= 0, \\
u_1(x) &= 0.75x, \\
u_2(x) &= 0.8906x, \\
u_3(x) &= 0.9483x, \\
u_4(x) &= 0.9748x, \\
\vdots
\end{align*}
\]

This gives the exact solution by \( u(x) = x \)

2. \( u(x) = x^2 - \frac{1}{6}x + \int_0^1 xt^2(t) \, dt \)

Differentiating both sides of this equation with respect to \( x \) yields

\[
\frac{du}{dx} = 2x - \frac{1}{6} + \int_0^1 tu^2(t) \, dt, \quad u(0) = 0.
\]

The correction functional for this equation is given by

\[
u_{n+1}(x) = u_n(x) - \int_0^x \left( \frac{u_n'(t) - 2t - \frac{1}{6} - \int_0^1 ru_n^2(r) \, dr}{dt} \right) \, dt
\]

This gives the following successive approximations
7.3. Nonlinear Fredholm Integral Equations of the First Kind

This gives the exact solution by \( u(x) = x^2 \)

3. \( u(x) = x^2 - \frac{1}{8}x + \int_0^1 xt^2u(t)\,dt \)

Differentiating both sides of this equation with respect to \( x \) and proceeding as before, we obtain the following successive approximations

\[
\begin{align*}
u_0(x) &= 0, \\
u_1(x) &= x^3 - 0.1250x, \\
u_2(x) &= x^3 - 0.0377x, \\
u_3(x) &= x^3 - 0.0122x, \\
u_4(x) &= x^3 - 0.0040x, \\
\vdots &. \\
\end{align*}
\]

This gives the exact solution by \( u(x) = x^3 \)

4. \( u(x) = x - \frac{1}{5}x^2 + \int_0^1 x^2t^2u(t)\,dt \)

Differentiating both sides of this equation with respect to \( x \) and proceeding as before, we obtain the exact solution by \( u(x) = x + x^2 \)

5. \( u(x) = x + \frac{34}{105}x^2 + \int_0^1 x^2t^2u(t)\,dt \)

Differentiating both sides of this equation with respect to \( x \) and proceeding as before, we obtain the exact solution by \( u(x) = x \).

7.3 Nonlinear Fredholm Integral Equations of the First Kind

Exercises 7.3

1. We set the transformation

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Chapter 7. Nonlinear Fredholm Integral Equations

\[ v(x) = u^2(x), u(x) = \sqrt{v(x)} \]

This gives

\[ v_\epsilon(x) = \frac{1}{\epsilon}(\frac{49}{60} - \alpha)x \]

where

\[ \alpha = \int_0^1 tv_\epsilon(t) \, dt \]

Proceeding as in the examples, we find

\[ \alpha = \frac{49}{60(1+3\epsilon)} \]

This gives

\[ u(x) = \frac{x}{\sqrt{20}} \sqrt{x} \]

Also

\[ u(x) = x + x^2 \]

2. We set the transformation

\[ v(x) = u^3(x), u(x) = \sqrt[3]{v(x)} \]

This gives

\[ v_\epsilon(x) = \frac{1}{\epsilon}(\frac{1}{120} - \alpha)x^2 \]

where

\[ \alpha = \int_0^1 t^2 v_\epsilon(t) \, dt \]

Proceeding as in the examples, we find

\[ \alpha = \frac{1}{120(1+5\epsilon)} \]

This gives
7.3. *Nonlinear Fredholm Integral Equations of the First Kind*

\[ u(x) = \sqrt[3]{\frac{x^2}{24}} \]

Also
\[ u(x) = x - x^3 \]

3. We set the transformation
\[ v(x) = u^3(x), \quad u(x) = \sqrt[3]{v(x)} \]

This gives
\[ v_\epsilon(x) = \frac{1}{\epsilon}(1 - \alpha)e^{2x} \]

where
\[ \alpha = \int_0^1 e^{-6t} v_\epsilon(t) \, dt \]

Proceeding as in the examples, we find
\[ \alpha = \frac{1 - e^{-4}}{1 - e^{-4} + 4\epsilon} \]

This gives
\[ u(x) = \sqrt[3]{\frac{4e^{2x}}{1 - e^{-x}}} \]

Also
\[ u(x) = e^{2x} \]

4. We set the transformation
\[ v(x) = u^4(x), \quad u(x) = \sqrt[4]{v(x)} \]

This gives
\[ v_\epsilon(x) = \frac{1}{\epsilon}(1 - \alpha)e^{x} \]

where
\[ \alpha = \int_0^1 e^{-4t} v_\epsilon(t) \, dt \]
Chapter 7. Nonlinear Fredholm Integral Equations

Proceeding as in the examples, we find
\[
\alpha = \frac{1-e^{-3}}{1-e^{-3}+3\epsilon}
\]
This gives
\[
u(x) = \sqrt{\frac{e^x}{1-e^{-3}}}
\]
Also
\[
u(x) = e^x
\]
5. We set the transformation
\[
v(x) = u^2(x), u(x) = \sqrt{v(x)}
\]
This gives
\[
v_\epsilon(x) = \frac{1}{\epsilon} (\frac{1}{4} - \alpha) x^2
\]
where
\[
\alpha = \int_0^1 t v_\epsilon(t) \, dt
\]
Proceeding as in the examples, we find
\[
\alpha = \frac{1}{4(1+4\epsilon)}
\]
This gives
\[
u(x) = x
\]
Also
\[
u(x) = \ln x
\]
6. We set the transformation
\[
v(x) = u^2(x), u(x) = \sqrt{v(x)}
\]
### 7.4. Weakly-Singular Nonlinear Fredholm Integral Equations

This gives

\[ v_{\epsilon}(x) = \frac{1}{\epsilon} \left( \frac{2}{125} - \alpha \right) x^2 \]

where

\[ \alpha = \int_0^1 t^2 v_{\epsilon}(t) \, dt \]

Proceeding as in the examples, we find

\[ \alpha = \frac{2}{125(1+5\epsilon)} \]

This gives

\[ u(x) = \frac{\sqrt{2}}{5} x \]

Also

\[ u(x) = x \ln x \]

### 7.4 Weakly-Singular Nonlinear Fredholm Integral Equations

#### Exercises 7.4

1. We decompose \( f(x) \) into

\[ f_0(x) = x \]
\[ f_1(x) = -\frac{16}{15} x^2 + \frac{2}{5} \sqrt{x - 1}(1 + \frac{4}{3} x + \frac{8}{3} x^2) \]

Consequently, we set the modified recurrence relation as

\[ u_0(x) = x \]
\[ u_1(x) = f_1(x) + \int_0^1 \frac{u_2(t)}{\sqrt{|x-t|}} \, dt = 0 \]

The exact solution is \( u(x) = x \).
2. We decompose $f(x)$ into

$$f_0(x) = x$$
$$f_1(x) = -\frac{32}{35}x^2 + \frac{2}{7}\sqrt{x-1}(1 + 6x + 8x^2 + 16x^3)$$

Consequently, we set the modified recurrence relation as

$$u_0(x) = x$$

$$u_1 x = f_1(x) + \frac{1}{\sqrt{|x-t|}} \int_0^1 u^2(t) \, dt = 0$$

The exact solution is $u(x) = x$.

3. We decompose $f(x)$ into

$$f_0(x) = 1 - x$$
$$f_1(x) = -2\sqrt{x}(1 - \frac{4}{5}x + \frac{8}{15}x^2) + \frac{16}{15}\sqrt{x-1}(1 - 2x + x^2)$$

Consequently, we set the modified recurrence relation as

$$u_0(x) = 1 - x$$

$$u_1 x = f_1(x) + \frac{1}{\sqrt{|x-t|}} \int_0^1 u^2(t) \, dt = 0$$

The exact solution is $u(x) = x$.

4. We decompose $f(x)$ into

$$f_0(x) = 1 + x$$
$$f_1(x) = -\frac{3}{2}x^\frac{3}{2}(1 + \frac{6}{5}x + \frac{9}{20}x^2) + \frac{1}{45}(x - 1)^\frac{3}{2}(123 + 90x + 27x^2)$$

Consequently, we set the modified recurrence relation as

$$u_0(x) = 1 + x$$

$$u_1 x = f_1(x) + \frac{1}{\sqrt{|x-t|}} \int_0^1 u^2(t) \, dt = 0$$

The exact solution is $u(x) = 1 + x$.
5. We decompose $f(x)$, then we use

\[
    u_0(x) = \sqrt{\cos x}
\]

\[
    u_1x = f_1(x) + \int_0^{\pi} \frac{u^*(t)}{(|\sin x - \sin t|)\pi} \, dt = 0
\]

The exact solution is $u(x) = \sqrt{\cos x}$.

6. We decompose $f(x)$, then we use

\[
    u_0(x) = \sqrt{\sin x}
\]

\[
    u_1x = f_1(x) + \int_0^{\pi} \frac{u^*(t)}{(|\cos x - \cos t|)\pi} \, dt = 0
\]

The exact solution is $u(x) = \sqrt{\sin x}$. 

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Chapter 8

Nonlinear Volterra Integral Equations

8.2 Nonlinear Volterra Integral Equations

8.2.1 The Series Solution Method

Exercises 8.2.1

1. \( u(x) = x^2 + \frac{1}{10} x^5 - \frac{1}{2} \int_0^x u^2(t) dt \)

Substituting \( u(x) \) by the series

\[
u(x) = \sum_{n=0}^{\infty} a_n x^n
\]

into both sides of the equation leads to

\[
\sum_{n=0}^{\infty} a_n x^n = x^2 + \frac{1}{10} x^5 - \frac{1}{2} \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt
\]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \ n \geq 0 \) and equating the coefficients of like powers of \( x \) in both sides yields

\[
a_2 = 1, \quad a_k = 0, k \neq 2
\]
Accordingly, we find

\[ u(x) = x^2 \]

2. \[ u(x) = x^2 + \frac{1}{12}x^6 - \frac{1}{2} \int_0^x tu^2(t)dt \]

Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = x^2 + \frac{1}{12}x^6 - \frac{1}{2} \int_0^x t \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \ n \geq 0 \) and equating the coefficients of like powers of \( x \) in both sides yields

\[ a_2 = 1, \quad a_k = 0, \ k \neq 2 \]

Accordingly, we find

\[ u(x) = x^2 \]

3. \[ u(x) = 1 - x^2 - \frac{1}{3}x^3 + \int_0^x u^2(t)dt \]

Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = 1 - x^2 - \frac{1}{3}x^3 + \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \ n \geq 0 \) and equating the coefficients of like powers of \( x \) in both sides yields
8.2. Nonlinear Volterra Integral Equations

\[ a_0 = a_1 = 1, \quad a_k = 0, k \geq 2 \]

Accordingly, we find

\[ u(x) = 1 + x \]

4. \( u(x) = 1 - x + x^2 - \frac{2}{3}x^3 - \frac{1}{5}x^5 + \int_0^x u^2(t)dt \)

Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = 1 - x + x^2 - \frac{2}{3}x^3 - \frac{1}{5}x^5 + \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \quad n \geq 0 \) and equating the coefficients of like powers of \( x \) in both sides yields

\[ a_0 = a_2 = 1, \quad a_k = 0, k \neq 0, 2 \]

Accordingly, we find

\[ u(x) = 1 + x^2 \]

5. \( u(x) = x^2 + \frac{1}{14}x^7 - \frac{1}{2} \int_0^x u^3(t)dt \)

Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = x^2 + \frac{1}{14}x^7 - \frac{1}{2} \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^3 dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \quad n \geq 0 \) and equating the coefficients of like powers of

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Chapter 8. Nonlinear Volterra Integral Equations

$x$ in both sides yields

\[ a_2 = 1, \quad a_k = 0, k \neq 2 \]

Accordingly, we find

\[ u(x) = x^2 \]

6. \[ u(x) = \frac{1}{2} + e^{-x} - \frac{1}{2}e^{-2x} - \int_0^x u^2(t)dt \]

Substituting $u(x)$ by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = \frac{1}{2} + e^{-x} - \frac{1}{2}e^{-2x} - \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form $t^n, n \geq 0$, using the Taylor expansions of the exponential functions and equating the coefficients of like powers of $x$ in both sides yields

\[ a_k = \frac{(-1)^k}{k!}, \quad k = 0, 1, 2, \ldots \]

Accordingly, we find

\[ u(x) = e^{-x} \]

7. \[ u(x) = 1 - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4 + \int_0^x u^3(t)dt \]

Substituting $u(x)$ by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = 1 - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4 + \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^3 dt \]
8.2. Nonlinear Volterra Integral Equations

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \ n \geq 0 \) and equating the coefficients of like powers of \( x \) in both sides yields

\[
a_0 = a_1 = 1, \quad a_k = 0, k \geq 2
\]

Accordingly, we find

\[
u(x) = 1 + x
\]

8. \( u(x) = \sin x - \frac{1}{2} x + \frac{1}{4} \sin 2x + \int_0^x u^2(t)dt\)

Substituting \( u(x) \) by the series

\[
u(x) = \sum_{n=0}^{\infty} a_n x^n
\]

into both sides of the equation leads to

\[
\sum_{n=0}^{\infty} a_n x^n = \sin x - \frac{1}{2} x + \frac{1}{4} \sin 2x + \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt
\]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, \ n \geq 0 \), using the Taylor series expansion of the trigonometric functions and equating the coefficients of like powers of \( x \) in both sides yields

\[
a_0 = 0
\]

\[
a_1 = 1
\]

\[
a_2 = 0
\]

\[
a_3 = -\frac{1}{3!}
\]

\[
a_{2k} = 0, \quad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!}, \ k = 0, 1, 2, \ldots
\]

Accordingly, we find

\[
u(x) = \sin x
\]
9. \( u(x) = \cos x - \frac{1}{2} x - \frac{1}{4} \sin 2x + \int_0^x u^2(t)dt \)

Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to

\[ \sum_{n=0}^{\infty} a_n x^n = \cos x - \frac{1}{2} x - \frac{1}{4} \sin 2x + \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt \]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, n \geq 0 \), using the Taylor series expansion of the trigonometric functions and equating the coefficients of like powers of \( x \) in both sides yields

\[ a_0 = 1 \]
\[ a_1 = 0 \]
\[ a_2 = -\frac{1}{2!} \]
\[ a_3 = 0 \]
\[ a_{2k} = (-1)^k \frac{1}{(2k)!}, \quad a_{2k+1} = 0, \quad k = 0, 1, 2, \ldots \]

Accordingly, we find

\[ u(x) = \cos x \]

10. \( u(x) = e^x + \frac{1}{2} x \left( e^{2x} - 1 \right) - \int_0^x x u^2(t)dt \)

Substituting \( u(x) \) by the series

\[ u(x) = \sum_{n=0}^{\infty} a_n x^n \]

into both sides of the equation leads to
8.2. Nonlinear Volterra Integral Equations

\[
\sum_{n=0}^{\infty} a_n x^n = e^x + \frac{1}{2} x (e^{2x} - 1) - \int_0^x x \left( \sum_{n=0}^{\infty} a_n t^n \right)^2 dt
\]

Evaluating the regular integrals at the right hand side that involve terms of the form \( t^n, n \geq 0 \), using the Taylor expansions of the exponential functions and equating the coefficients of like powers of \( x \) in both sides yields

\[ a_k = \frac{1}{k!}, \quad k = 0, 1, 2, \ldots \]

Accordingly, we find

\[ u(x) = e^x \]

8.2.2 The Adomian Decomposition Method

Exercises 8.2.2

1. \( u(x) = 3x + \frac{1}{24} x^4 - \frac{1}{18} \int_0^x (x - t) u^2(t) dt \)

   Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the procedure of the modified decomposition method discussed before, we set

   \[ u_0(x) = 3x \]

   Accordingly, the first component is given by

   \[ u_1(x) = \frac{1}{24} x^4 - \frac{1}{18} \int_0^x (x - t) u_0^2(t) dt \]

   so that

   \[ u_1(x) = 0 \]

   Consequently, the other components vanish, hence

   \[ u(x) = 3x \]

2. \( u(x) = 2x - \frac{1}{2} x^4 + \frac{1}{4} \int_0^x u^3(t) dt \)
Chapter 8. Nonlinear Volterra Integral Equations

Using the decomposition $u(x) = \sum_{n=0}^{\infty} u_n(x)$ into both sides and following the procedure of the modified decomposition method discussed before, we set

$$u_0(x) = 2x$$

Accordingly, the first component is given by

$$u_1(x) = -\frac{1}{2}x^4 + \frac{1}{4} \int_0^x u_0^2(t) \, dt$$

so that

$$u_1(x) = 0$$

Consequently, the other components vanish, hence

$$u(x) = 2x$$

3. $u(x) = \sin x + \frac{1}{8} \sin 2x - \frac{1}{4}x + \frac{1}{2} \int_0^x u_0^2(t) \, dt$

Using the decomposition $u(x) = \sum_{n=0}^{\infty} u_n(x)$ into both sides and following the procedure of the modified decomposition method discussed before, we set

$$u_0(x) = \sin x$$

Accordingly, the first component is given by

$$u_1(x) = \frac{1}{8} \sin 2x - \frac{1}{4}x + \frac{1}{2} \int_0^x u_0^2(t) \, dt$$

so that

$$u_1(x) = 0$$

Consequently, the other components vanish, hence

$$u(x) = \sin x$$

4. $u(x) = x^2 + \frac{1}{5}x^6 - \int_0^x u^2(t) \, dt$
8.2. Nonlinear Volterra Integral Equations

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the procedure of the modified decomposition method discussed before, we set

\[ u_0(x) = x^2 \]

Accordingly, the first component is given by

\[ u_1(x) = \frac{1}{5} x^5 - \int_0^x u_0^2(t) \, dt \]

so that

\[ u_1(x) = 0 \]

Consequently, the other components vanish, hence

\[ u(x) = x^2 \]

5. \( u(x) = x + \int_0^x (x - t) u^3(t) \, dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the Adomian decomposition method discussed before, we set

\[ u_0(x) = x \]

Accordingly, the first components are given by

\[ u_1(x) = \frac{1}{20} x^5 \]
\[ u_2(x) = \frac{1}{720} x^9 \]

Hence

\[ u(x) \approx x + \frac{1}{20} x^5 + \frac{1}{720} x^9 + \cdots \]

6. \( u(x) = 1 + \int_0^x (x - t)^2 u^2(t) \, dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the Adomian decomposition method discussed before, we set
Chapter 8. Nonlinear Volterra Integral Equations

\( u_0(x) = 1 \)

Accordingly, the first components are given by

\[
\begin{align*}
  u_1(x) &= \frac{1}{3} x^3 \\
  u_2(x) &= \frac{1}{90} x^6
\end{align*}
\]

Hence

\[ u(x) \approx x + \frac{1}{3} x^3 + \frac{1}{90} x^6 + \cdots \]

7. \( u(x) = 1 + \int_0^x (x - t)^2 u^3(t) dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the Adomian decomposition method discussed before, we set

\( u_0(x) = 1 \)

Accordingly, the first components are given by

\[
\begin{align*}
  u_1(x) &= \frac{1}{3} x^3 \\
  u_2(x) &= \frac{1}{60} x^6
\end{align*}
\]

Hence

\[ u(x) \approx x + \frac{1}{3} x^3 + \frac{1}{60} x^6 + \cdots \]

8. \( u(x) = x + \int_0^x (x - t)^2 u^2(t) dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the Adomian decomposition method discussed before, we set

\( u_0(x) = x \)

Accordingly, the first components are given by

\( u_1(x) = \frac{1}{6} x^5 \)

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8.2. Nonlinear Volterra Integral Equations

\[ u_2(x) = \frac{1}{756}x^9 \]

Hence

\[ u(x) \approx x + \frac{1}{6}x^5 + \frac{1}{756}x^9 + \cdots \]

9. \( u(x) = 1 + \int_0^x (t + u^2(t)) \, dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the Adomian decomposition method discussed before, we set \( u_0(x) = 1 \)

Accordingly, the first components are given by

\[ u_1(x) = 2x \]
\[ u_2(x) = \frac{5}{2}x^2 \]

Hence

\[ u(x) \approx 1 + 2x + \frac{5}{2}x^2 + \cdots \]

10. \( u(x) = 1 + \int_0^x (t^2 + u^2(t)) \, dt \)

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the Adomian decomposition method discussed before, we set \( u_0(x) = 1 \)

Accordingly, the first components are given by

\[ u_1(x) = x + \frac{1}{4}x^3 \]
\[ u_2(x) = x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 \]

Hence

\[ u(x) \approx 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \cdots \]

11. \( u(x) = \sec x + \tan x + x - \int_0^x (1 + u^2(t)) \, dt, \quad x < \frac{\pi}{2} \)
Chapter 8. Nonlinear Volterra Integral Equations

Using the decomposition \( u(x) = \sum_{n=0}^{\infty} u_n(x) \) into both sides and following the procedure of the modified decomposition method discussed before, we set
\[
u_0(x) = \sec x
\]

Accordingly, the first component is given by
\[
u_1(x) = 0
\]

Consequently, the other components vanish, hence
\[u(x) = \sec x\]

12. \( u(x) = \tan x - \frac{1}{4} \sin 2x - \frac{1}{2} x + \int_0^x \frac{1}{1 + u^2(t)} \, dt, \quad x < \frac{\pi}{2}\)

Using the decomposition
\[u(x) = \sum_{n=0}^{\infty} u_n(x)\]

into both sides and following the procedure of the modified decomposition method discussed before, we set
\[
u_0(x) = \tan x
\]

Accordingly, the first component is given by
\[
u_1(x) = 0
\]

Consequently, the other components vanish, hence
\[u(x) = \tan x\]

8.2.3 The Variational Iteration Method

Exercises 8.2.3

1. \( u(x) = x - \frac{1}{20} x^5 + \int_0^x (x - t) u^3(t) \, dt \)

Differentiating both sides of this equation, and using Leibniz rule, we find

www.MathSchoolinternational.com
8.2. Nonlinear Volterra Integral Equations

\[ u'(x) = 1 - \frac{1}{4}x^4 + \int_0^x u^3(t) \, dt, \, u(0) = 0 \]

The correction functional is

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(\xi) - 1 + \frac{1}{4}\xi^4 - \int_0^x u^3_n(r) \, dr \right) d\xi \]

This will give the following successive approximations

\[
\begin{align*}
    u_0(x) &= 0, \\
    u_1(x) &= x - \frac{1}{20}x^5, \\
    u_2(x) &= 2x - \frac{1}{20}x^5 + \left( \frac{1}{8}x^5 - \frac{1}{480}x^9 \right) + \frac{1}{20800}x^{13} + \cdots, \\
    u_3(x) &= 2x - \frac{1}{480}x^9 + \left( \frac{1}{120}x^9 + \frac{1}{20800}x^{13} \right) - \frac{1}{20800}x^{13} + \cdots,
\end{align*}
\]

Cancelling the noise terms gives the exact solution by \( u(x) = x \).

2. \( u(x) = x^2 - \frac{1}{56}x^8 + \int_0^x (x - t)u^3(t) \, dt \)

Differentiating both sides of this equation, and using Leibniz rule, we find

\[ u'(x) = 2x - \frac{1}{7}x^7 + \int_0^x u^3(t) \, dt, \, u(0) = 0 \]

The correction functional is

\[ u_{n+1}(x) = u_n(x) - \int_0^x \left( u'_n(\xi) - 2\xi + \frac{1}{7}\xi^7 - \int_0^x u^3_n(r) \, dr \right) d\xi \]

This will give the following successive approximations

\[
\begin{align*}
    u_0(x) &= 0, \\
    u_1(x) &= x^2 - \frac{1}{56}x^8, \\
    u_2(x) &= x^2 - \frac{1}{10192}x^{14} + \cdots, \\
    u_3(x) &= x^2 + \cdots,
\end{align*}
\]

Cancelling the noise terms gives the exact solution by \( u(x) = x^2 \).

3. \( u(x) = x + x^2 - \frac{1}{12}x^4 - \frac{1}{16}x^5 - \frac{1}{35}x^6 + \int_0^x (x - t)u^2(t) \, dt \)

Proceeding as before gives the following successive approximations
Chapter 8. Nonlinear Volterra Integral Equations

\[ u_0(x) = 0, \]
\[ u_1(x) = x + x^2 - \frac{1}{12}x^4 - \frac{1}{15}x^5 - \frac{1}{30}x^6, \]
\[ u_2(x) = x + x^2 + \text{noise terms}, \]
\[ \vdots \]

Cancelling the noise terms gives the exact solution by \( u(x) = x + x^2. \)

4. \[ u(x) = 1 + x - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \int_0^x (x-t)u^2(t)dt \]

Proceeding as before and by cancelling the noise terms, we obtain the exact solution by \( u(x) = 1 + x. \)

5. \[ u(x) = 1 + x - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \int_0^x (x-t)^2u^2(t)dt \]

Proceeding as before and by cancelling the noise terms, we obtain the exact solution by \( u(x) = 1 + x. \)

8.3 Nonlinear Volterra Integral Equations of the First Kind

8.3.1 The Series Solution Method

Exercises 8.3.1

1. Using the Taylor series of the left side, and substituting the series form of \( u(x) \) yields

\[ \frac{1}{12}x^4 - \frac{1}{30}x^5 + \cdots = \int_0^x (x-t) \left[ a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots \right]^2 dt \]

Integrating the right side, and equating the coefficients of like powers of \( x \)

\[ a_0 = 0, \quad a_1 = \pm 1, \quad a_2 = 0, \quad a_3 = \pm \frac{1}{3!}, \cdots \]

Consequently, the exact solution is given by

\[ u(x) = \pm \sin x \]

2. Using the Taylor series of the left side, and substituting the series form of \( u(x) \) yields

\[ \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \cdots = \int_0^x (x-t) \left[ a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots \right]^2 dt \]
8.3. **Nonlinear Volterra Integral Equations of the First Kind**

Integrating the right side, and equating the coefficients of like powers of \( x \)

\[
a_0 = \pm 1, \quad a_1 = \pm 1, \quad a_2 = \pm \frac{1}{2}, \quad a_3 = \pm \frac{1}{3!}, \cdots
\]

Consequently, the exact solution is given by

\[
u(x) = \pm e^x
\]

3. Using the Taylor series of the left side, and substituting the series form of \( u(x) \) yields

\[
x + \frac{3}{2} x^2 + \frac{3}{8} x^3 + \frac{9}{80} x^4 + \cdots = \int_0^x \left[ a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \right] dt
\]

Integrating the right side, and equating the coefficients of like powers of \( x \)

\[
a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3!}, \cdots
\]

Consequently, the exact solution is given by

\[
u(x) = \pm e^x
\]

4. Using the Taylor series of the left side, and substituting the series form of \( u(x) \) yields

\[
\frac{1}{2} x^2 - \frac{3}{2} x^3 + \frac{1}{18} x^5 - \frac{2}{315} x^7 + \cdots = \int_0^x (x-t) \left[ a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \right] dt
\]

Integrating the right side, and equating the coefficients of like powers of \( x \)

\[
a_0 = \pm 1, \quad a_1 = \mp 1, \quad a_2 = \mp \frac{1}{2}, \quad a_3 = \pm \frac{1}{3!}, \cdots
\]

Consequently, the exact solution is given by

\[
u(x) = \pm (\cos x - \sin x)
\]
Chapter 8. Nonlinear Volterra Integral Equations

8.3.2 Conversion to a Volterra Equation of the Second Kind

Exercises 8.3.2

1. Differentiating both sides, and using Leibniz rule gives

\[ v(x) = x^6 + \frac{1}{7}x^7 - \int_0^x v(t) \, dt \]

This gives

\[ v_0(x) = x^6, \quad v_1(x) = 0 \]

This in turn gives

\[ u(x) = \pm x^3 \]

2. Differentiating both sides, and using Leibniz rule gives

\[ v(x) = x^8 + \frac{1}{9}x^9 - \int_0^x v(t) \, dt \]

This gives

\[ v_0(x) = x^8, \quad v_1(x) = 0 \]

This in turn gives

\[ u(x) = \pm x^4 \]

3. Differentiating both sides, and using Leibniz rule gives

\[ v(x) = x^{12} + \frac{1}{13}x^{14} - \int_0^x v(t) \, dt \]

This gives

\[ v_0(x) = x^{12}, \quad v_1(x) = 0 \]

This in turn gives

\[ u(x) = \pm x^4 \]

4. Differentiating both sides, and using Leibniz rule gives
8.4 Nonlinear Weakly-Singular Volterra Equation

\[ v(x) = \frac{3}{2} e^{2x} - \frac{1}{2} - \int_0^x v(t) \, dt \]

This gives

\[ v_0(x) = e^{2x}, \quad v_1(x) = 0 \]

This in turn gives

\[ u(x) = \pm e^x \]

5. Proceeding as before gives

\[ v_0(x) = x^2 e^{2x}, \quad v_1(x) = 0 \]

This in turn gives

\[ u(x) = \pm xe^x \]

5. Proceeding as before gives

\[ v_0(x) = x^2 e^{2x}, \quad v_1(x) = 0 \]

This in turn gives

\[ u(x) = \pm xe^x \]

6. Proceeding as before gives

\[ v_0(x) = e^{-2x}, \quad v_1(x) = 0 \]

This in turn gives

\[ u(x) = \pm e^{-x} \]

8.4 Nonlinear Weakly-Singular Volterra Equation

Exercises 8.4

1. Decompose \( f(x) \) into two parts, then set:
Chapter 8. Nonlinear Volterra Integral Equations

\[ u_0(x) = x^3, \quad u_1(x) = -\frac{729}{2048} x^\frac{25}{3} + \int_0^x \frac{A_0(t)}{(x^2 - t^2)^\frac{1}{3}} \, dt = 0 \]

This gives

\[ u(x) = x^3 \]

2. Decompose \( f(x) \) into two parts, then set:

\[ u_0(x) = x^4, \quad u_1(x) = -\frac{231}{2048} \pi x^{12} + \int_0^x \frac{u^3(t)}{\sqrt{x^2 - t^2}} \, dt = 0 \]

This gives

\[ u(x) = x^4 \]

3. Decompose \( f(x) \) into two parts, then set:

\[ u_0(x) = \sqrt[3]{1 + x^4}, \quad u_1(x) = -\frac{\pi}{2} (1 + \frac{3}{8} x^4) + \int_0^x \frac{u^5(t)}{\sqrt{x^2 - t^2}} \, dt = 0 \]

This gives

\[ u(x) = \sqrt[3]{1 + x^4} \]

4. Decompose \( f(x) \) into two parts, then set:

\[ u_0(x) = \sqrt[3]{1 + x + x^3}, \quad u_1(x) = -(\frac{\pi}{2} + x + \frac{2}{3} x^3) + \int_0^x \frac{u^5(t)}{\sqrt{x^2 - t^2}} \, dt = 0 \]

This gives

\[ u(x) = \sqrt[3]{1 + x + x^3} \]

5. Decompose \( f(x) \) into two parts, then set:

\[ u_0(x) = \sqrt{\cos x}, \quad u_1(x) = -2 \sqrt{\sin x} + \int_0^x \frac{u^5(t)}{\sqrt{\sin x - \sin t}} \, dt = 0 \]

This gives

\[ u(x) = \sqrt{\cos x} \]

6. Decompose \( f(x) \) into two parts, then set:

\[ u_0(x) = \sqrt{\sin x}, \quad u_1(x) = 2 \sqrt{\cos x - 1} + \int_0^x \frac{u^5(t)}{\sqrt{\cos x - \cos t}} \, dt = 0 \]

This gives

\[ u(x) = \sqrt{\sin x} \]

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